Learning in the Context of Set Theoretic Estimation: an Efficient and Unifying Framework for Adaptive Machine Learning and Signal Processing

Sergios Theodoridis[1]

a joint work with
K. Slavakis (Univ. of Peloponnese, Greece), and
I. Yamada (Tokyo Institute of Technology, Japan)

[1]University of Athens, Athens, Greece.

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April 11th, 2012

"ΟΥΔΕΙΣ ΑΓΕΩΜΕΤΡΗΤΟΣ ΕΙΣΙΤΩ"

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(“Those who do not know geometry are not welcome here”)

Plato’s Academy of Philosophy

Part A

Outline of Part A

- The set theoretic estimation approach and multiple intersecting closed convex sets.
- The fundamental tool of metric projections in Hilbert spaces.
- Online classification and regression.
- The concept of Reproducing Kernel Hilbert Spaces (RKHS) and nonlinear processing.
- Distributive learning in sensor networks.

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Special Cases

Smoothing, prediction, curve-fitting, regression, classification, filtering, system identification, and beamforming.

The More Classical Approach

Select a loss function $\mathcal{L}(\cdot,\cdot)$ and estimate $f(\cdot)$ so that

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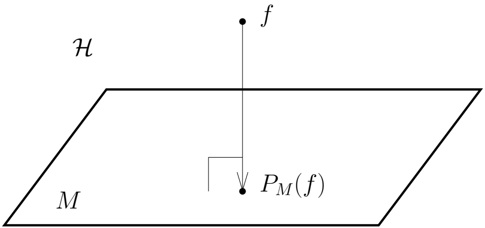
- The existence of a-priori information in the form of constraints makes the task even more difficult.
- The optimization task is solved iteratively, and iterations freeze after a finite number of steps. Thus, the obtained solution lies in a neighborhood of the optimal one.
- The stochastic nature of the data and the existence of noise add another uncertainty to the optimality of the obtained solution.

- In this talk, we are concerned in finding a set of solutions, which are in agreement with all the available information.
- This will be achieved in the general context of
  - Set theoretic estimation.
  - Convexity.
  - Mappings or operators, e.g., projections, and their associated fixed point sets.

Projection onto a Closed Subspace

Theorem

Given a Euclidean $\mathbb{R}^m$ or a Hilbert space $\mathcal{H}$, the projection of a point f onto a closed subspace M is the unique point $P_M(f) \in M$ that lies closest to f (Pythagoras Theorem).



Projection onto a Closed Convex Set

Theorem

Let C be a closed convex set in a Hilbert space $\mathcal{H}$. Then, for each $f \in \mathcal{H}$, there exists a unique $f_* \in C$ such that

$$\|f - f_*\| = \min_{g \in C} \|f - g\| =: d(f, C).$$

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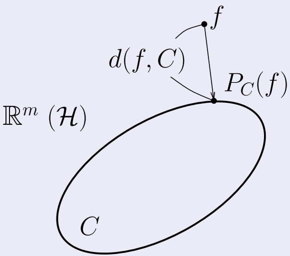
Definition (Metric Projection Mapping)

The projection is the mapping $P_C : \mathcal{H} \to C : f \mapsto P_C(f) := f_*$.



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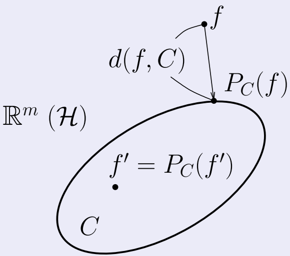
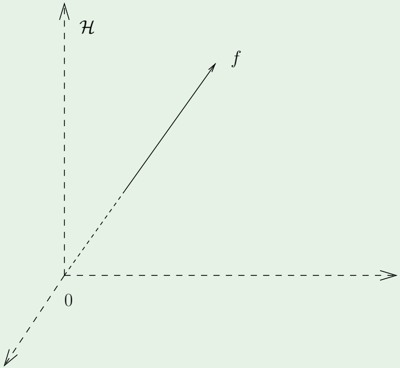


Projection Mappings

Example (Hyperplane $H := \{g \in \mathcal{H} : \langle g, a \rangle = c\}$)



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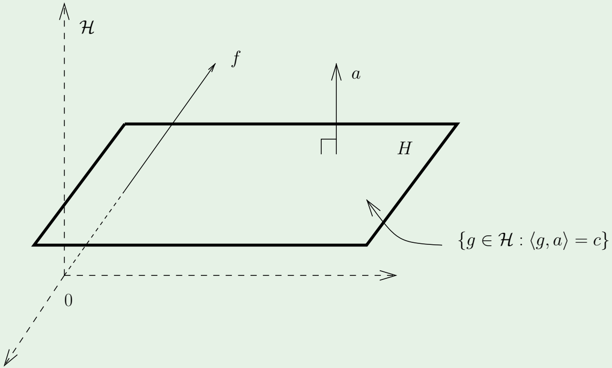

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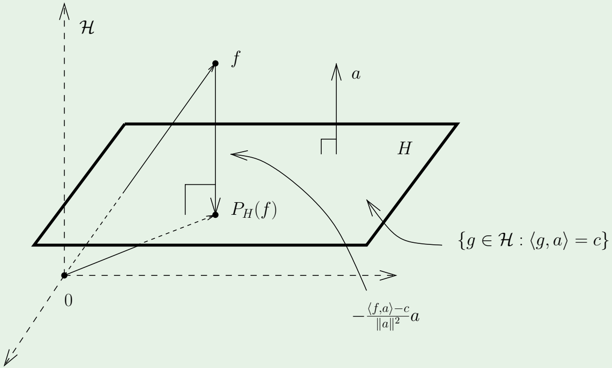

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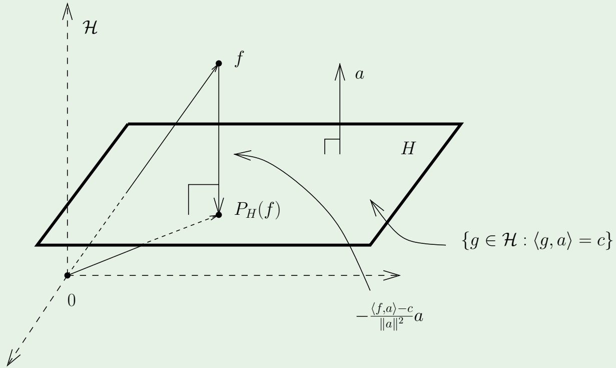
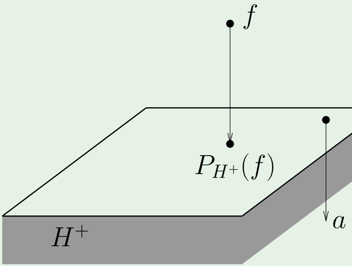


$$P_H(f) = f - \frac{\langle f, a\rangle - c}{\|a\|^2} a, \qquad \forall f \in \mathcal{H}.$$

Example (Halfspace $H^+ := \{g \in \mathcal{H} : \langle g, a \rangle \geq c\}$)



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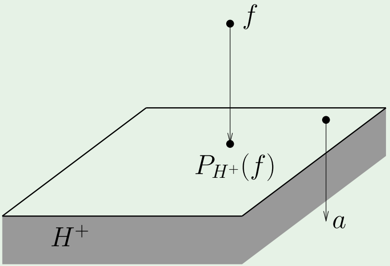
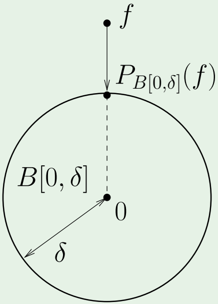


$$P_{H^+}(f) = f - \frac{\min\left\{0, \langle f, a \rangle - c\right\}}{\|a\|^2} a, \qquad \forall f \in \mathcal{H}.$$

Example (Closed Ball $B[0,\delta] := \{g \in \mathcal{H} : \|g\| \leq \delta\}$)



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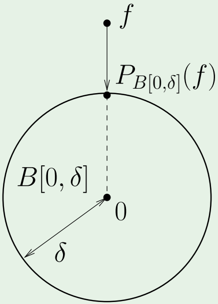
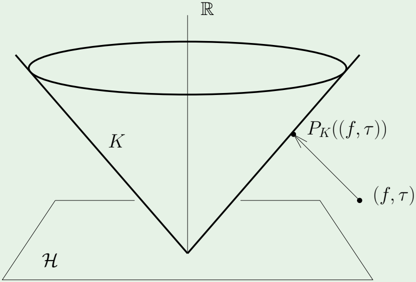


$$P_{B[0,\delta]}(f) := \frac{\delta}{\max\{\delta, \|f\|\}} f, \quad \forall f \in \mathcal{H}.$$

Example (Icecream Cone $K := \left\{(f,\tau) \in \mathcal{H} \times \mathbb{R} : \|f\| \geq \tau\right\}$)



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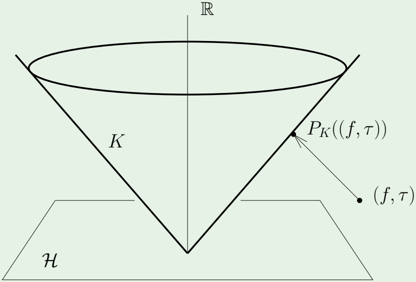
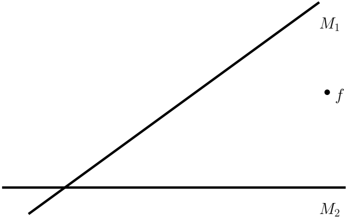


$$P_K((f,\tau)) = \begin{cases} (f,\tau), & \text{if } \|f\| \leq \tau, \\ (0,0), & \text{if } \|f\| \leq -\tau, \\ \frac{\|f\|+\tau}{2}\left(\frac{f}{\|f\|}, 1\right), & \text{otherwise}, \end{cases} \qquad \forall (f,\tau) \in \mathcal{H} \times \mathbb{R}.$$

Alternating Projections

Composition of Projection Mappings: Let M_1 and M_2 be closed subspaces in the Hilbert space $\mathcal{H}$. For any $f \in \mathcal{H}$, define the sequence of projections:



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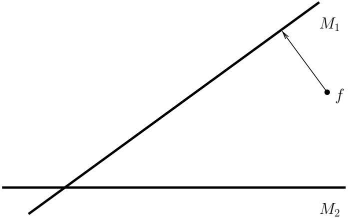
$$P_{M_1}(f).$$



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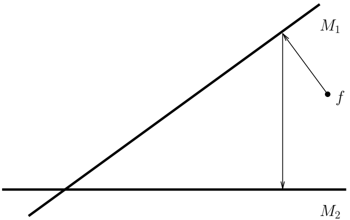

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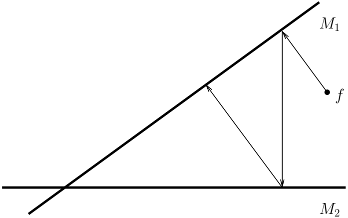

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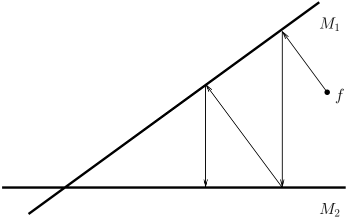

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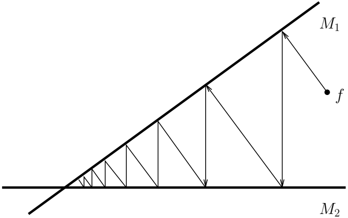

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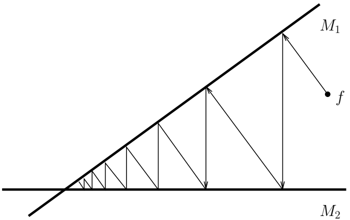

Theorem ([von Neumann '33])

For any $f \in \mathcal{H}$, $\lim_{n\to\infty} \left(P_{M_2} P_{M_1}\right)^n (f) = P_{M_1 \cap M_2}(f)$.

Projections Onto Convex Sets (POCS)

Theorem (POCS[1])

Given a finite number of closed convex sets $C_1, \ldots, C_p$, with $\bigcap_{i=1}^p C_i \neq \emptyset$, let their associated projection mappings be $P_{C_1}, \ldots, P_{C_p}$. For any $f_0 \in \mathcal{H}$, this defines the sequence of points

$$f_{n+1} := P_{C_p} \cdots P_{C_1}(f_n), \qquad \forall n,$$

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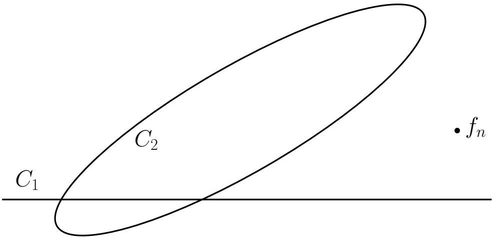

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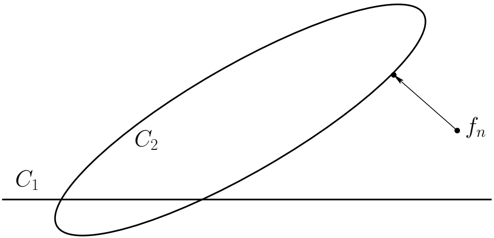

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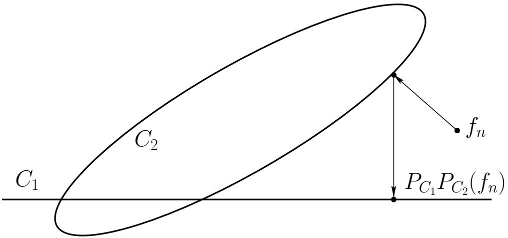

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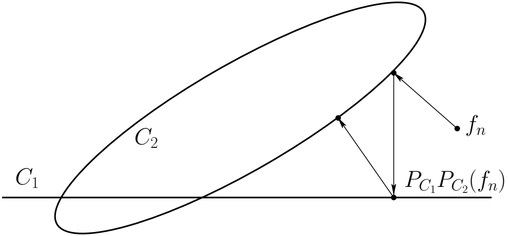

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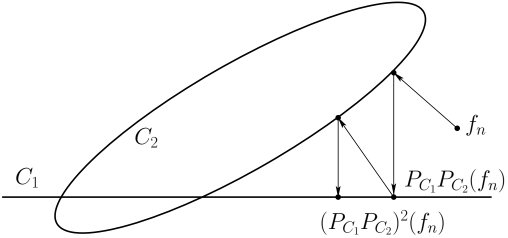

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Extrapolated Parallel Projection Method (EPPM)

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$$f_{n+1} = f_n + \mu_n \Bigg(\underbrace{\sum_{i=1}^{p} w_i P_{C_i}(f_n)}_{\text{Convex combination of projections}} - f_n \Bigg), \quad \forall n,$$

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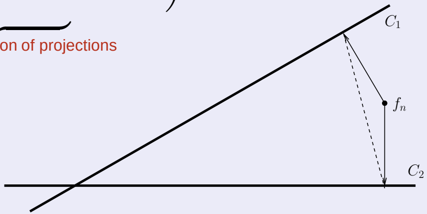

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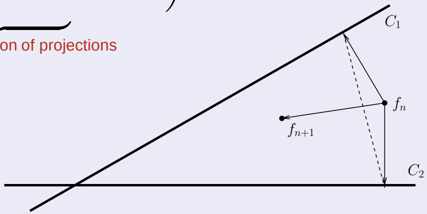

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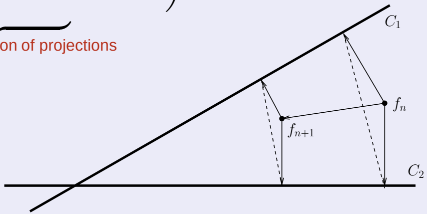

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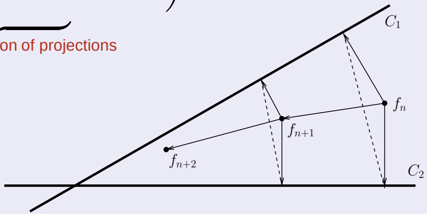

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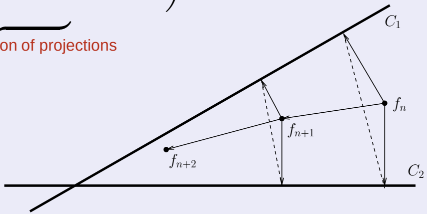
$$f_{n+1} = f_n + \mu_n \Bigg(\underbrace{\sum_{i=1}^{p} w_i P_{C_i}(f_n)}_{\text{Convex combination of projections}} - f_n \Bigg), \quad \forall n,$$

converges weakly to a point f_* in $\bigcap_{i=1}^p C_i$, where $\mu_n \in (\epsilon, \mathcal{M}_n)$, for $\epsilon \in (0,1)$, and

$$\mathcal{M}_n := \frac{\sum_{i=1}^p w_i \left\| P_{C_i}(f_n) - f_n \right\|^2}{\left\| \sum_{i=1}^p w_i P_{C_i}(f_n) - f_n \right\|^2}.$$



[2] [Pierra '84].

Infinite Number of Closed Convex Sets

Adaptive Projected Subgradient Method (APSM)[3]

Given an infinite number of closed convex sets $(C_n)_{n\geq 0}$, let their associated projection mappings be $(P_{C_n})_{n\geq 0}$. For any starting point f_0, and an integer $q > 0$, let the sequence

$$f_{n+1} = f_n + \mu_n \left(\sum_{j=n-q+1}^{n} w_j P_{C_j}(f_n) - f_n \right), \quad \forall n,$$

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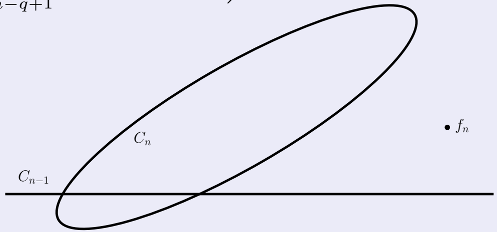

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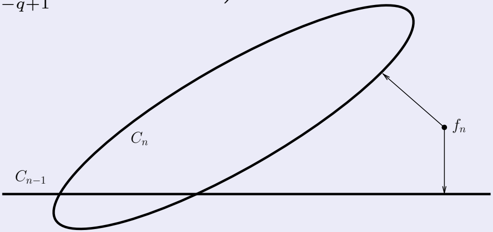

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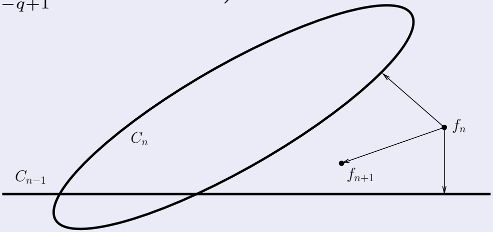

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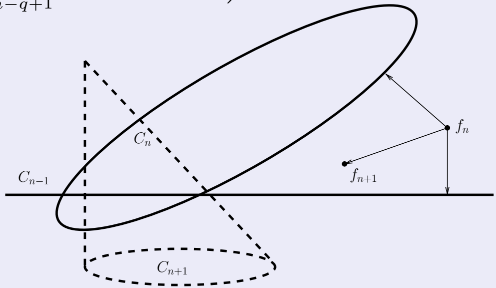

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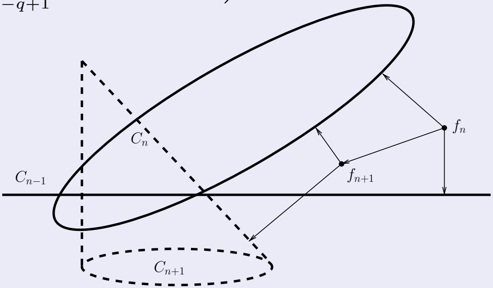

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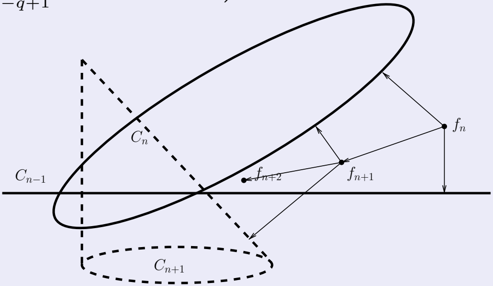

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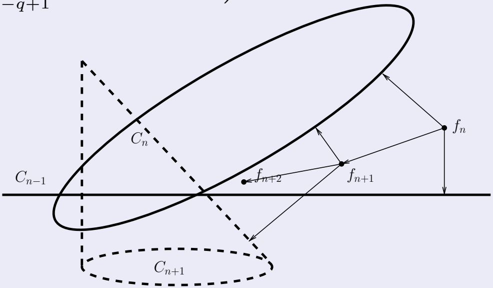
where $\mu_n \in (0, 2\mathcal{M}_n)$, and $\mathcal{M}_n := \frac{\sum_{j=n-q+1}^{n} w_j \left\| P_{C_j}(f_n) - f_n \right\|^2}{\left\| \sum_{j=n-q+1}^{n} w_j P_{C_j}(f_n) - f_n \right\|^2}$.

Under certain constraints the above sequence converges strongly to a point $f_* \in \mathrm{clos}\big(\bigcup_{m\geq 0} \bigcap_{n\geq m} C_n\big)$.



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The Task

Given a set of training samples $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N \subset \mathbb{R}^m$ and a set of corresponding desired responses $y_0, \ldots, y_N$, estimate a function $f(\cdot) : \mathbb{R}^m \to \mathbb{R}$ that fits the data.

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The Expected / Empirical Risk Function approach

Estimate f so that the expected risk based on a loss function $\mathcal{L}(\cdot, \cdot)$ is minimized:

$$\min_f \mathsf{E}\left\{\mathcal{L}(f(\boldsymbol{x}), y)\right\},$$

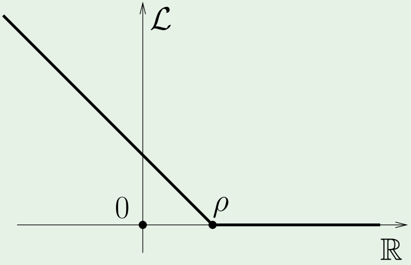
or, in practice, the empirical risk is minimized:

$$\min_f \sum_{n=0}^{N} \mathcal{L}\big(f(\boldsymbol{x}_n), y_n\big).$$

Example (Classification)

For a given margin $\rho \geq 0$, and $y_n \in \{+1, -1\}$, $\forall n$, define the soft margin loss function:

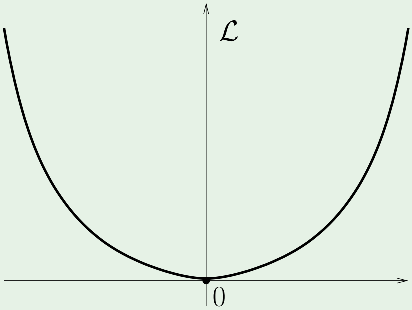
$$\mathcal{L}\big(f(\boldsymbol{x}_n), y_n\big) := \max\big\{0, \rho - y_n f(\boldsymbol{x}_n)\big\}, \qquad \forall n.$$



Example (Regression)

The square loss function:

$$\mathcal{L}\big(f(\boldsymbol{x}_n), y_n\big) := \big(y_n - f(\boldsymbol{x}_n)\big)^2, \qquad \forall n.$$



The Set Theoretic Estimation Approach

Main Idea

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- The intersection of all these sets constitutes the family of solutions.
- The family of solutions is known as the feasibility set.

That is, represent each cost and constraint by an equivalent set C_n and find the solution

$$f \in \bigcap_n C_n \subset \mathcal{H}.$$

Classification: The Soft Margin Loss

The Setting

Let the training data set $(\boldsymbol{x}_n, y_n) \subset \mathbb{R}^m \times \{+1, -1\}$, $n = 0, 1, \ldots$.
Assume the two class task,

$$\begin{cases} y_n = +1, & \boldsymbol{x}_n \in W_1, \\ y_n = -1, & \boldsymbol{x}_n \in W_2. \end{cases}$$

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$$\text{Find} \quad f(\boldsymbol{x}) = \boldsymbol{\theta}^t \boldsymbol{x} + b, \quad \text{so that}$$

$$\begin{cases} \boldsymbol{\theta}^t \boldsymbol{x}_n + b \geq \rho, & \text{if } y_n = +1, \\ \boldsymbol{\theta}^t \boldsymbol{x}_n + b \leq \rho, & \text{if } y_n = -1. \end{cases}$$

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Find all those $\boldsymbol{\theta}$ so that $\quad y_n \boldsymbol{\theta}^t \boldsymbol{x}_n \geq \rho, \quad n = 0, 1, \ldots$

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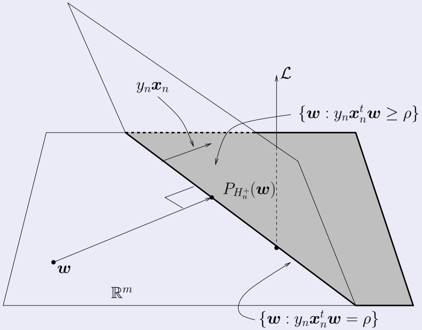
The Equivalent Set

$$H_n^+ := \left\{ \boldsymbol{\theta} \in \mathbb{R}^m : y_n \boldsymbol{x}_n^t \boldsymbol{\theta} \geq \rho \right\}, \, n = 0, 1, \ldots.$$



The feasibility set

For each pair $(\boldsymbol{x}_n, y_n)$, form the equivalent halfspace H_n^+, and

$$\text{find} \quad \boldsymbol{\theta}_* \in \bigcap_n H_n^+.$$

If linearly separable, the problem is feasible.

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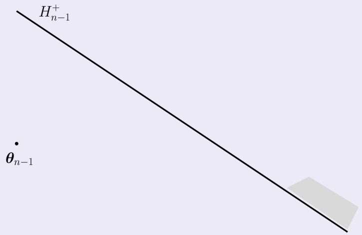

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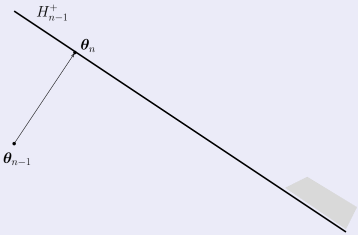

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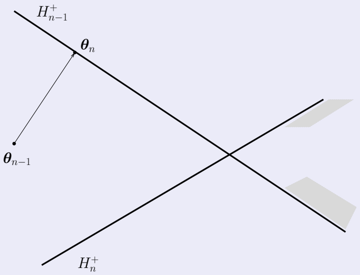

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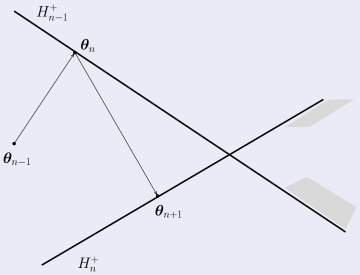

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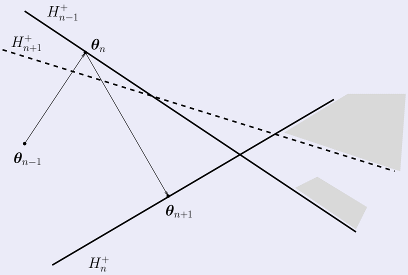

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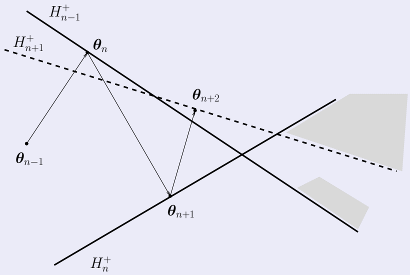

Algorithmic Solution to Online Classification

$$\boldsymbol{\theta}_{n+1} := \boldsymbol{\theta}_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{H_n^+}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n \right),$$

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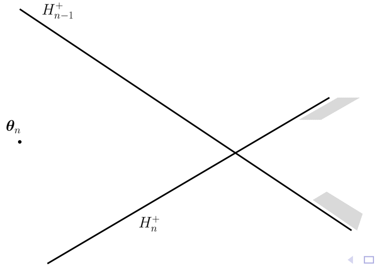

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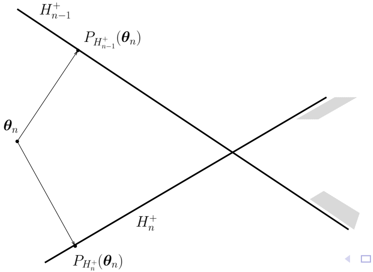

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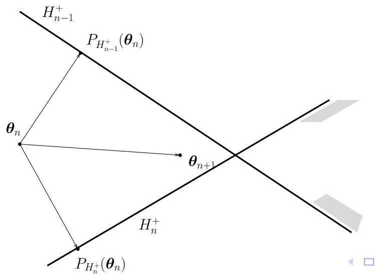

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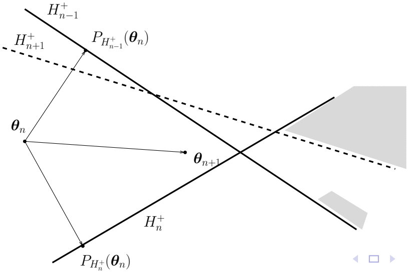

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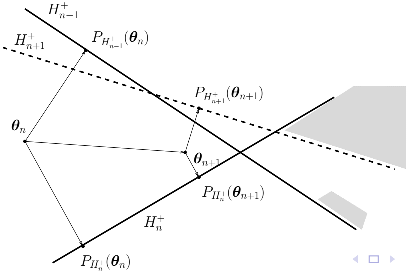

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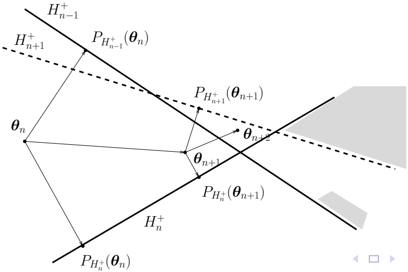
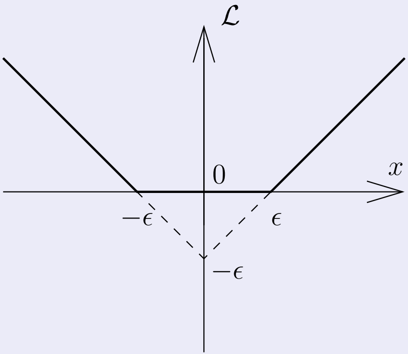


The linear ϵ-insensitive loss function case

$$\mathcal{L}(x) := \max\{0, |x| - \epsilon\}, \quad x \in \mathbb{R}.$$



The Piece of Information

Given $(\boldsymbol{x}_n, y_n) \in \mathbb{R}^m \times \mathbb{R}$, find $\boldsymbol{\theta} \in \mathbb{R}^m$ such that

$$\left|\boldsymbol{\theta}^t \boldsymbol{x}_n - y_n\right| \leq \epsilon, \qquad \forall n.$$

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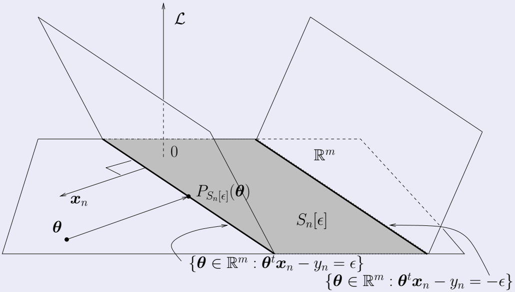
The Equivalent Set (Hyperslab)

$$S_n[\epsilon] := \left\{\boldsymbol{\theta} \in \mathbb{R}^m : |\boldsymbol{\theta}^t \boldsymbol{x}_n - y_n| \leq \epsilon\right\}, \quad \forall n.$$



Projection onto a Hyperslab

$$P_{S_n[\epsilon]}(\boldsymbol{\theta}) = \boldsymbol{\theta} + \beta \boldsymbol{x}_n, \quad \forall \boldsymbol{\theta} \in \mathbb{R}^m,$$

where

$$\beta := \begin{cases} \frac{y_n - \boldsymbol{\theta}^t \boldsymbol{x}_n - \epsilon}{\boldsymbol{x}_n^t \boldsymbol{x}_n}, & \text{if } \boldsymbol{\theta}^t \boldsymbol{x}_n - y_n < -\epsilon, \\ 0, & \text{if } \left| \boldsymbol{\theta}^t \boldsymbol{x}_n - y_n \right| \leq \epsilon, \\ -\frac{\boldsymbol{\theta}^t \boldsymbol{x}_n - y_n - \epsilon}{\boldsymbol{x}_n^t \boldsymbol{x}_n}, & \text{if } \boldsymbol{\theta}^t \boldsymbol{x}_n - y_n > \epsilon. \end{cases}$$

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The feasibility set

For each pair $(\boldsymbol{x}_n, y_n)$, form the equivalent hyperslab S_n, and

$$\text{find} \quad \boldsymbol{\theta}_* \in \bigcap_n S_n[\epsilon].$$

Algorithm for the Online Regression

Assume weights $\omega_j^{(n)} \geq 0$ such that $\sum_{j=n-q+1}^{n} \omega_j^{(n)} = 1$. For any $\boldsymbol{\theta}_0 \in \mathbb{R}^m$,

$$\boldsymbol{\theta}_{n+1} := \boldsymbol{\theta}_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n \right), \quad \forall n \geq 0,$$

where the extrapolation coefficient $\mu_n \in (0, 2\mathcal{M}_n)$ with

$$\mathcal{M}_n := \begin{cases} \frac{\sum_{j=n-q+1}^{n} \omega_j^{(n)} \|P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\|^2}{\|\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\|^2}, & \text{if } \sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) \neq \boldsymbol{\theta}_n, \\ 1, & \text{otherwise.} \end{cases}$$

Geometry of the Algorithm

$\boldsymbol{\theta}_n$

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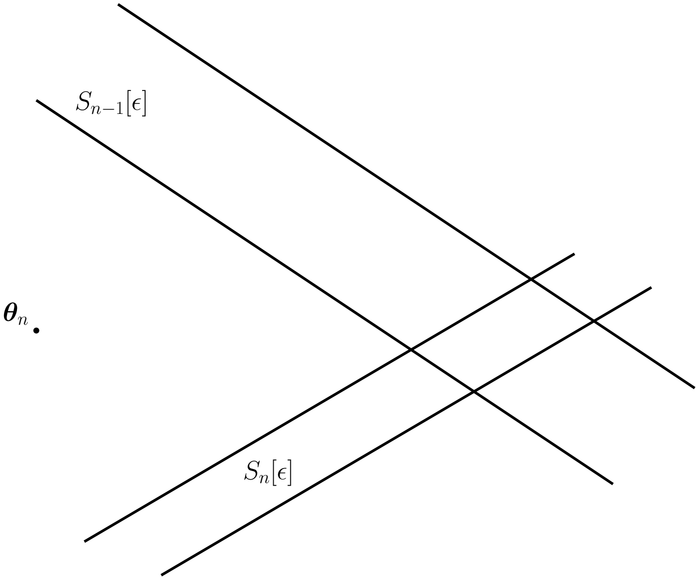

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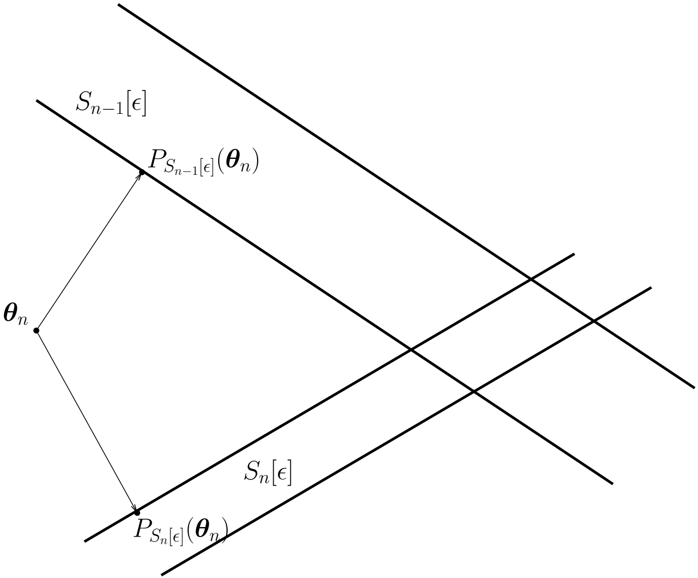

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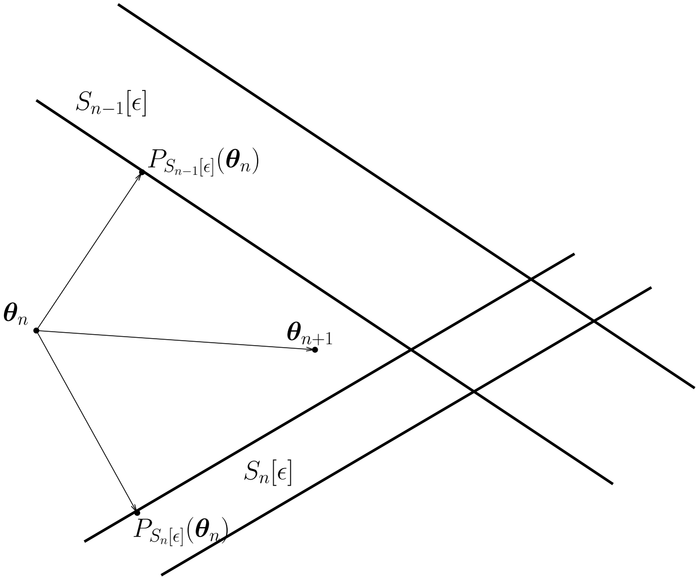

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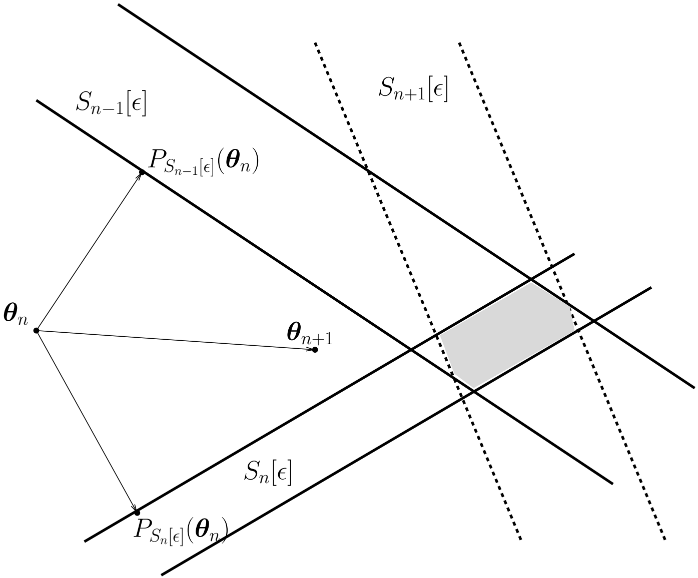

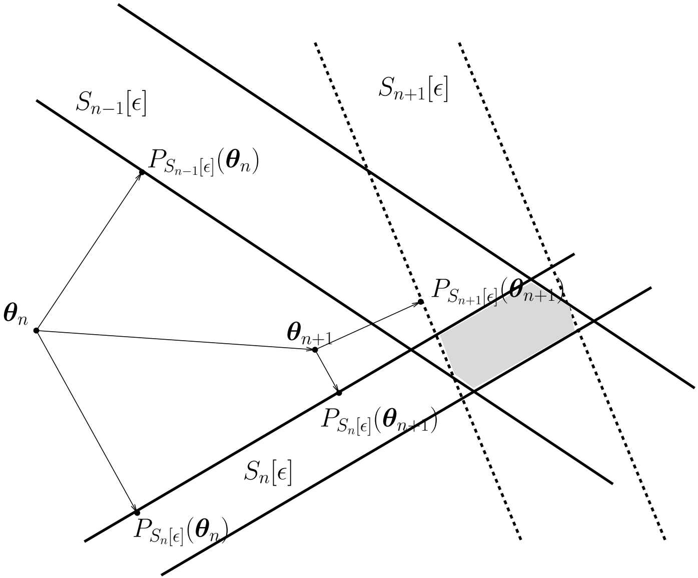

$S_{n-1}[\epsilon]$
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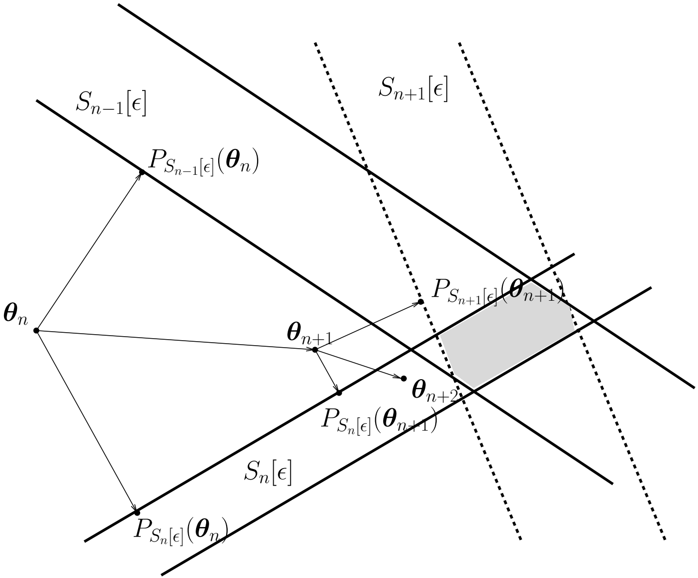

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Definition

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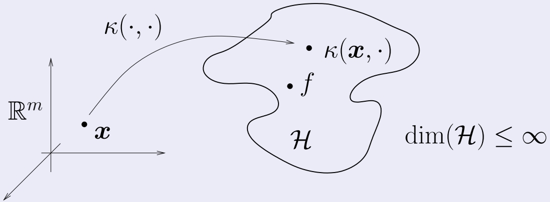
Then $\mathcal{H}$ is called a Reproducing Kernel Hilbert Space (RKHS).



Properties of the Kernel Function

- If such a kernel function exists, then it is a symmetric and positive definite kernel; for any real numbers $a_0, a_1, \ldots, a_N$, any $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots \boldsymbol{x}_N \in \mathbb{R}^m$, and any N,

$$\sum_{i=0}^{N} \sum_{j=0}^{N} a_i a_j \kappa(\boldsymbol{x}_i, \boldsymbol{x}_j) \geq 0.$$

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- The reverse is also true. Let

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  be symmetric and positive definite. Then, there exists an RKHS of functions on $\mathbb{R}^m$, such that $\kappa(\cdot, \cdot)$ is a reproducing kernel of $\mathcal{H}$.
- Each RKHS is uniquely defined by a $\kappa(\cdot, \cdot)$, and each (symmetric) positive definite kernel, $\kappa(\cdot, \cdot)$, uniquely defines an RKHS[4].

---

[4] [Aronszajn '50]

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- This is an important property since it leads to an easy, black box rule, which transforms a nonlinear task to a linear one; this is done by the following steps...

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- Use an algorithm that can be casted (modified) in terms of inner products.
- Replace inner product computations with kernel ones:

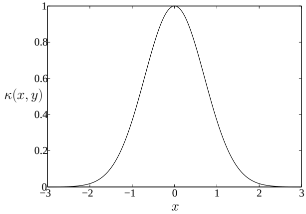
$$\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = \kappa(\boldsymbol{x}, \boldsymbol{y}).$$

  This is the step that brings the nonlinearity in the modeling.

Kernel Functions Examples

- The Gaussian kernel:

$$\kappa(\boldsymbol{x}, \boldsymbol{y}) := \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{\sigma^2}\right),$$



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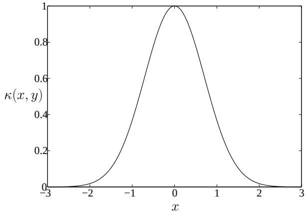
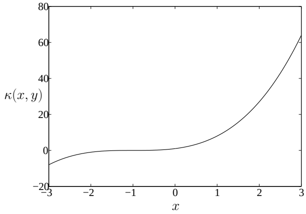


- The polynomial kernel:

$$\kappa(\boldsymbol{x}, \boldsymbol{y}) := \left(\boldsymbol{x}^t \boldsymbol{y} + 1\right)^d,$$



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- Then, the solution of the task

$$\min_{f \in \mathcal{H}} \sum_{n=0}^{N} \mathcal{L}(y_n, f(\boldsymbol{x}_n)) + \Omega(\|f\|),$$

admits a representation of the form:

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Example

$$\mathcal{L}(y_n, f(\boldsymbol{x}_n)) := \big(y_n - f(\boldsymbol{x}_n)\big)^2,$$
$$\Omega(\|f\|) := \|f\|^2 = \langle f, f \rangle.$$

The Goal

Let the training data set $(\boldsymbol{x}_n, y_n) \subset \mathbb{R}^m \times \mathbb{R}$, $n = 0, 1, \ldots$.

- $\boldsymbol{x}_n \mapsto \kappa(\boldsymbol{x}_n, \cdot)$, which is a function of one variable.

Regression in RKHS

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- $\boldsymbol{x}_n \mapsto \kappa(\boldsymbol{x}_n, \cdot)$, which is a function of one variable.
- Find $f \in \mathcal{H}$ such that

$$|f(\boldsymbol{x}_n) - y_n| \leq \epsilon, \qquad \forall n.$$

Set Theoretic Estimation Approach to Regression

The Piece of Information

Given $(\boldsymbol{x}_n, y_n) \in \mathbb{R}^m \times \mathbb{R}$, $n = 0, 1, 2, \ldots$, find $f \in \mathcal{H}$ such that

$$|\langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n| \leq \epsilon, \qquad \forall n.$$

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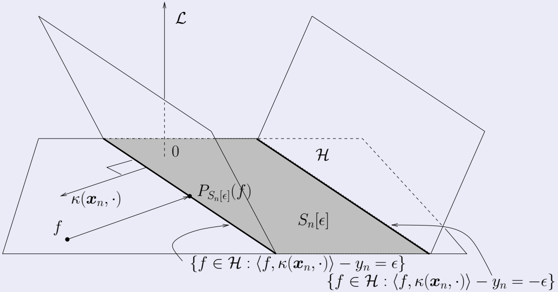

Projection onto a Hyperslab

$$P_{S_n[\epsilon]}(f) = f + \beta\kappa(\boldsymbol{x}_n, \cdot), \forall f \in \mathcal{H},$$

where

$$\beta := \begin{cases} \frac{y_n - \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - \epsilon}{\kappa(\boldsymbol{x}_n, \boldsymbol{x_n})}, & \text{if } \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n < -\epsilon, \\ 0, & \text{if } |\langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n| \leq \epsilon, \\ -\frac{\langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n - \epsilon}{\kappa(\boldsymbol{x}_n, \boldsymbol{x_n})}, & \text{if } \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n > \epsilon. \end{cases}$$

Projection onto a Hyperslab

$$P_{S_n[\epsilon]}(f) = f + \beta\kappa(\boldsymbol{x}_n, \cdot), \forall f \in \mathcal{H},$$

where

$$\beta := \begin{cases} \frac{y_n - \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - \epsilon}{\kappa(\boldsymbol{x}_n, \boldsymbol{x}_n)}, & \text{if } \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n < -\epsilon, \\ 0, & \text{if } |\langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n| \leq \epsilon, \\ -\frac{\langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n - \epsilon}{\kappa(\boldsymbol{x}_n, \boldsymbol{x}_n)}, & \text{if } \langle f, \kappa(\boldsymbol{x}_n, \cdot)\rangle - y_n > \epsilon. \end{cases}$$

The feasibility set

For each pair $(\boldsymbol{x}_n, y_n)$, form the equivalent hyperslab S_n, and

$$\text{find} \quad f_* \in \bigcap_{n \geq n_0} S_n[\epsilon].$$

Algorithm for Online Regression in RKHS

For $f_0 \in \mathcal{H}$, execute the following algorithm[5]

$$f_{n+1} := f_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(f_n) - f_n \right), \quad \forall n \geq 0,$$

where the extrapolation coefficient $\mu_n \in (0, 2\mathcal{M}_n)$ with

$$\mathcal{M}_n := \begin{cases} \frac{\sum_{j=n-q+1}^{n} \omega_j^{(n)} \|P_{S_j[\epsilon]}(f_n) - f_n\|^2}{\|\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(f_n) - f_n\|^2}, & \text{if } \sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(f_n) \neq f_n, \\ 1, & \text{otherwise.} \end{cases}$$

[5] [Slavakis, Theodoridis, Yamada '09].

Sparsification

As time goes by:

$$f_n := \sum_{i=0}^{n-1} \gamma_i^{(n)} \kappa(\boldsymbol{x}_i, \cdot).$$

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$$\forall n \geq 0, \quad f_n \in B[0, \delta] := \{f \in \mathcal{H} : \|f\| \leq \delta\} : \text{ Closed Ball.}$$

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Goal

Thus, we are looking for a classifier $f \in \mathcal{H}$ such that

$$f \in B[0, \delta] \cap \Big(\bigcap_{n \geq n_0} S_n[\epsilon] \Big).$$

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Geometric Illustration of the Algorithm

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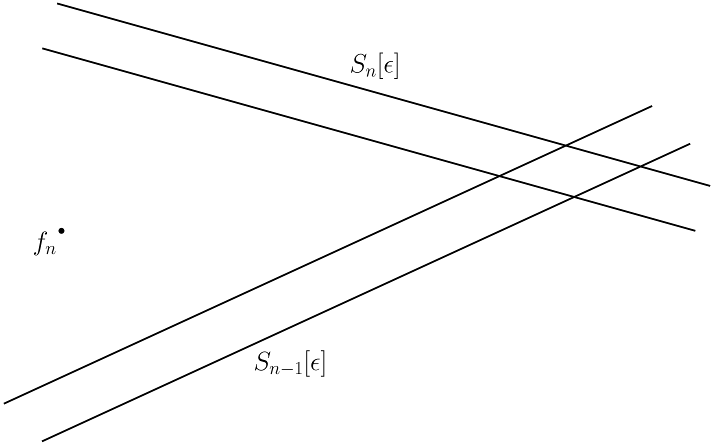

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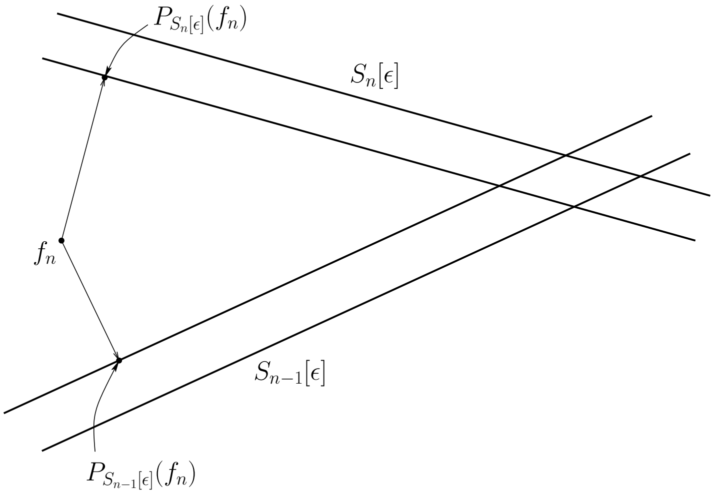

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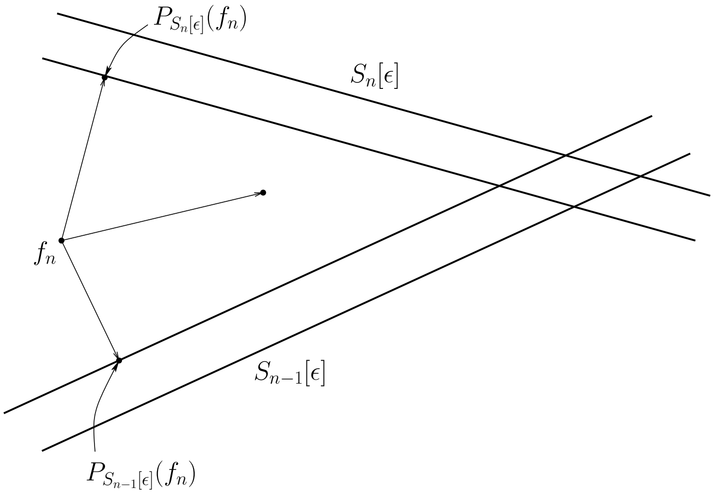

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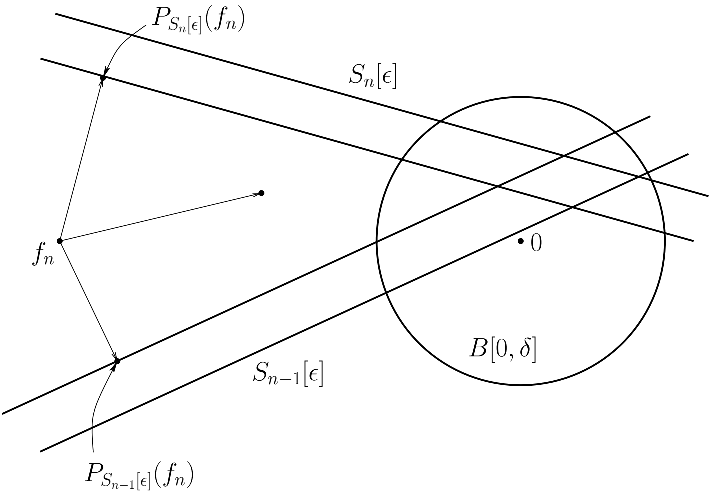

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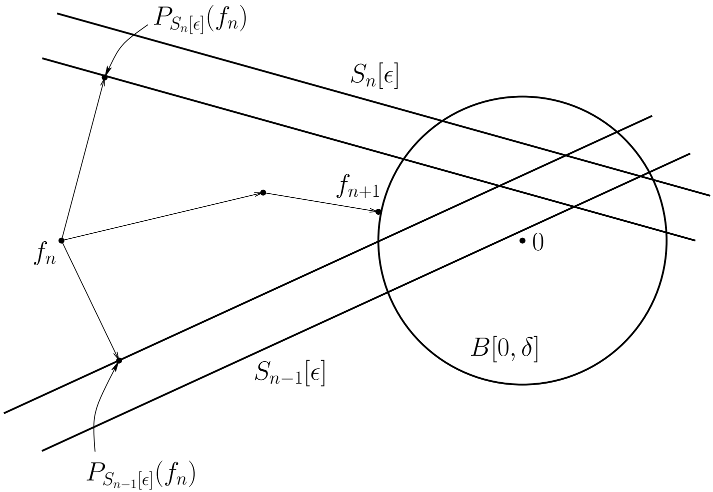

Distributive Learning for Sensor Networks

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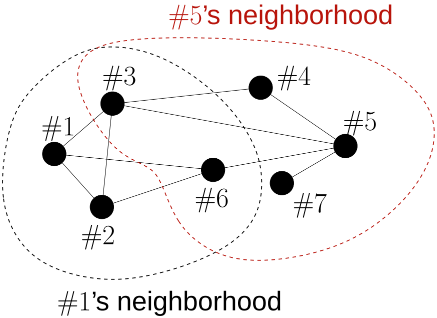
The goal is to drive the locally computed estimates to converge to the same value. This is known as consensus.

- The most commonly used topology is the diffusion network:



Problem Formulation

- Let a node set denoted as $\mathcal{N} := \{1, 2, \ldots, N\}$ and each node, k, at time, n, has access to the measurements

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we assume that there exists a linear system, $\boldsymbol{\theta}_*$, such that

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where $v_k(n)$ is the noise.
The task is to estimate the common $\boldsymbol{\theta}_*$.

The Algorithm (node k)

- Combine estimates received from the neighborhood $\mathcal{N}_k$:

$$\phi_k(n) := \sum_{l \in \mathcal{N}_k} c_{k,l}(n+1)\boldsymbol{\theta}_l(n).$$

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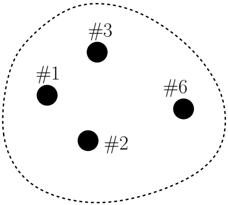
- Perform the adaptation step[7]:

$$\boldsymbol{\theta}_k(n+1) := \boldsymbol{\phi}_k(n) + \mu_k(n+1) \left(\sum_{j=n-q+1}^{n} \omega_{k,j} P_{S_{k,j}}(\boldsymbol{\phi}_k(n)) - \boldsymbol{\phi}_k(n) \right).$$

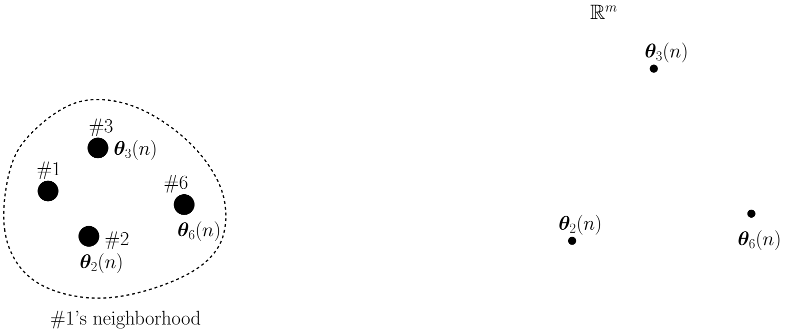
[7] [Chouvardas, Slavakis, Theodoridis, '11].

The Geometry of the Algorithm



#1's neighborhood

The Geometry of the Algorithm



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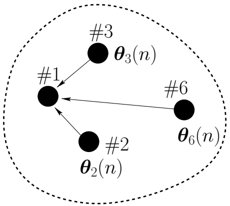

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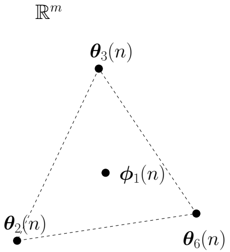

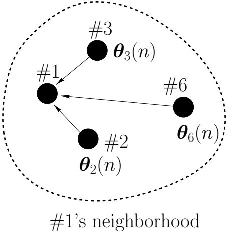
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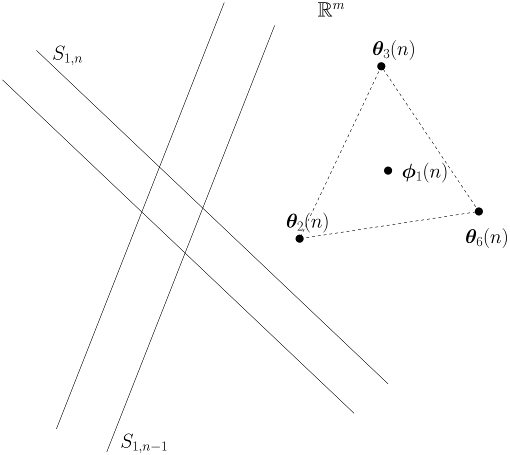
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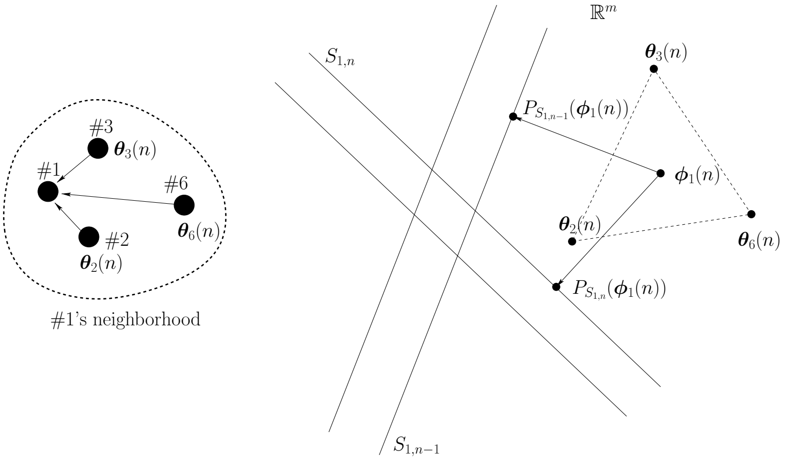
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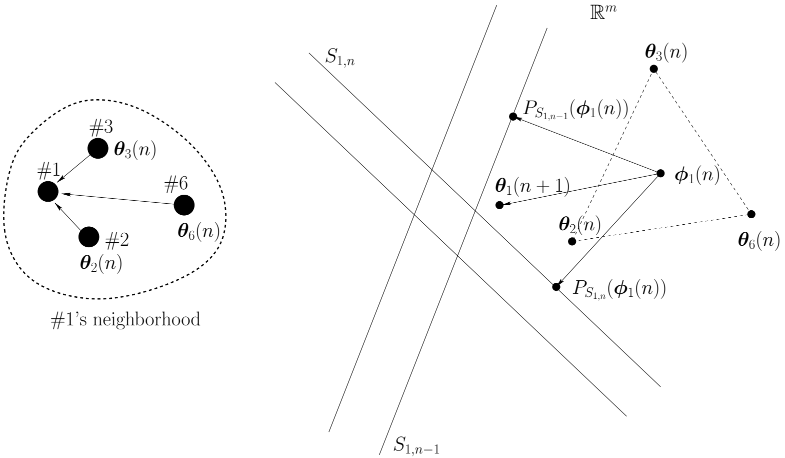

Part B

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  - Beamforming task.
  - Sparsity-aware learning problem.
- Our objective is to show that a large variety of constrained online learning tasks can be unified under a common umbrella; the Adaptive Projected Subgradient Method (APSM).

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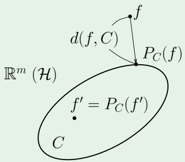
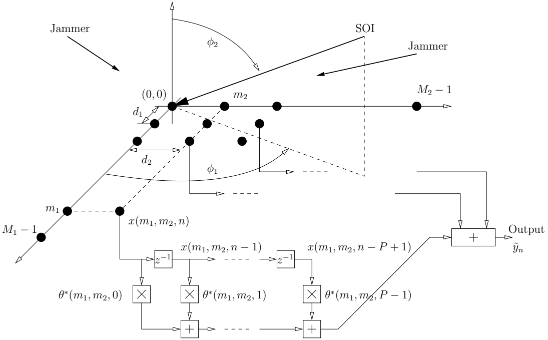
Example

If C is a closed convex set in $\mathcal{H}$, then $\mathrm{Fix}(P_C) = C$.



Beamforming

- Antenna arrays are vastly utilized for space-time filtering:



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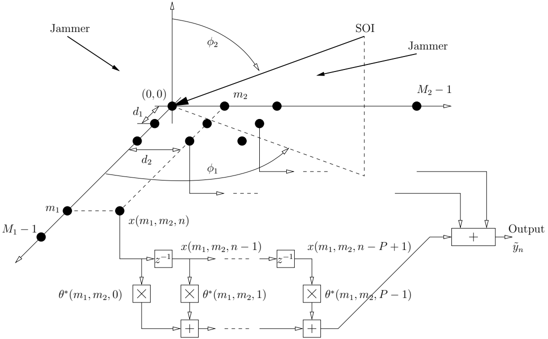

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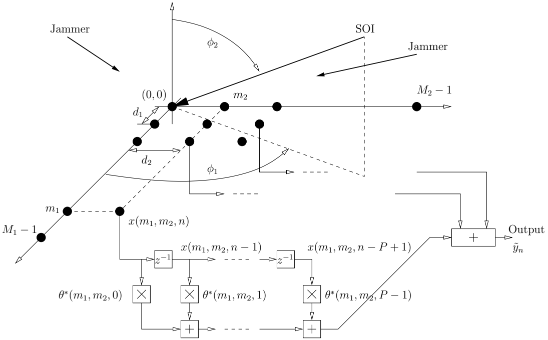

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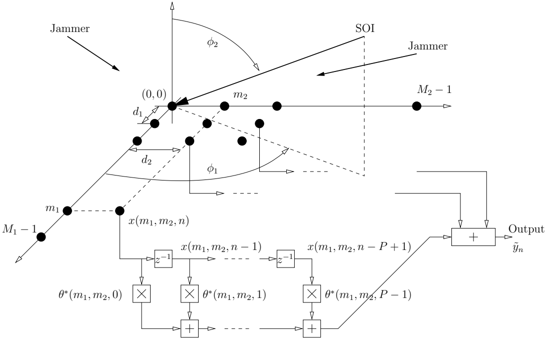

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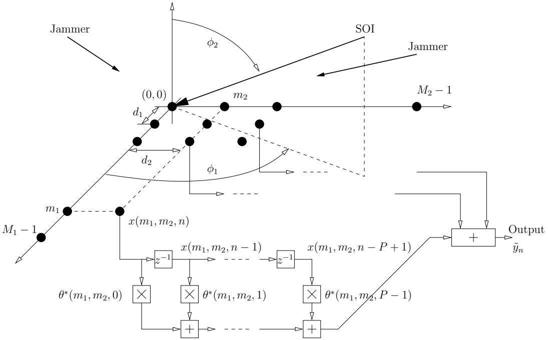

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The beamformer is the vector $\boldsymbol{\theta}$.

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By utilizing all the available a-priori knowledge, reconstruct the SOIs, while, in the meantime, suppress the jamming signals.

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Given the previous a-priori info, and the set of data $(y_n, \boldsymbol{x}_n)$, $n = 0, 1, 2, \ldots$, compute $\boldsymbol{\theta}$ such that

$$\boldsymbol{\theta}^t \boldsymbol{x}_n \approx y_n, \quad \forall n.$$

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Nulls

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Affinely Constrained Beamforming

A large variety of a-priori knowledge in beamforming problems can be cast by means of affine constraints; given a matrix C and a vector $\boldsymbol{g}$:

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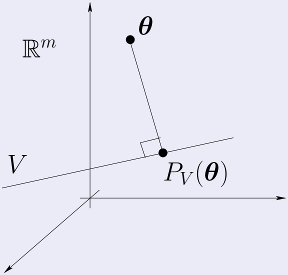
$$\forall\boldsymbol{\theta}, \quad \boldsymbol{C}^t\boldsymbol{\theta} \neq \boldsymbol{g}.$$

Projection onto the affine set V

Given $V := \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^m} \left\|C^t\boldsymbol{\theta} - \boldsymbol{g}\right\|$, the metric projection mapping onto V is given by

$$P_V(\boldsymbol{\theta}) = \boldsymbol{\theta} - C^{t\dagger}\left(C^t\boldsymbol{\theta} - \boldsymbol{g}\right), \quad \forall\boldsymbol{\theta} \in \mathbb{R}^m,$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse of a matrix.



Affinely Constrained Algorithm

- At time n, given the training data $(y_n, \boldsymbol{x}_n)$, define the hyperslab:

$$S_n[\epsilon] := \left\{ \boldsymbol{\theta} \in \mathbb{R}^m : \left| \boldsymbol{x}_n^t \boldsymbol{\theta} - y_n \right| \le \epsilon \right\}.$$

Affinely Constrained Algorithm

- At time n, given the training data $(y_n, \boldsymbol{x}_n)$, define the hyperslab:

$$S_n[\epsilon] := \left\{\boldsymbol{\theta} \in \mathbb{R}^m : \left|\boldsymbol{x}_n^t \boldsymbol{\theta} - y_n\right| \leq \epsilon\right\}.$$

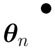
- For any initial point $\boldsymbol{\theta}_0$, and $\forall n$,

$$\boldsymbol{\theta}_{n+1} := P_V\left(\boldsymbol{\theta}_n + \mu_n \left(\sum_{i=n-q+1}^{n} \omega_i^{(n)} P_{S_i[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\right)\right),$$

$$\mu_n \in (0, 2\mathcal{M}_n),$$

$$\mathcal{M}_n := \begin{cases} \frac{\sum_{j=n-q+1}^{n} \omega_j^{(n)} \|P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\|^2}{\|\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\|^2}, & \text{if } \sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) \neq \boldsymbol{\theta}_n, \\ 1, & \text{otherwise.} \end{cases}$$

Geometry of the Algorithm



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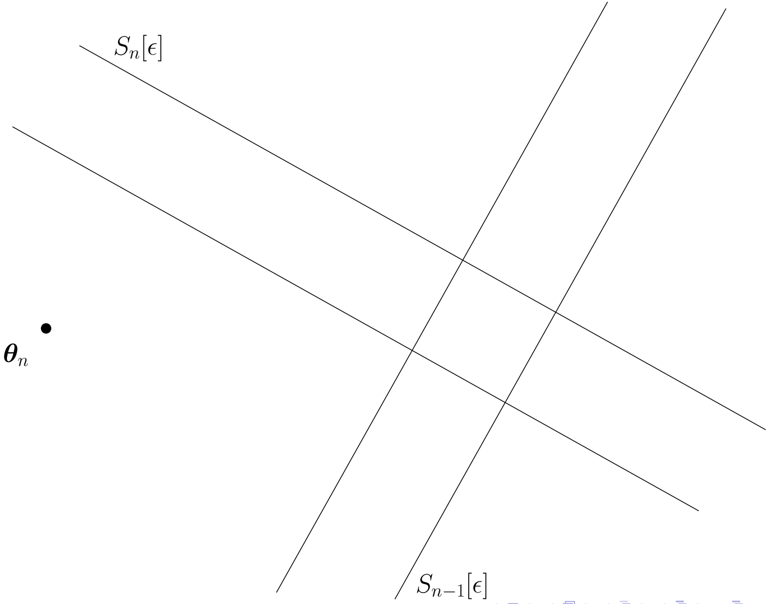

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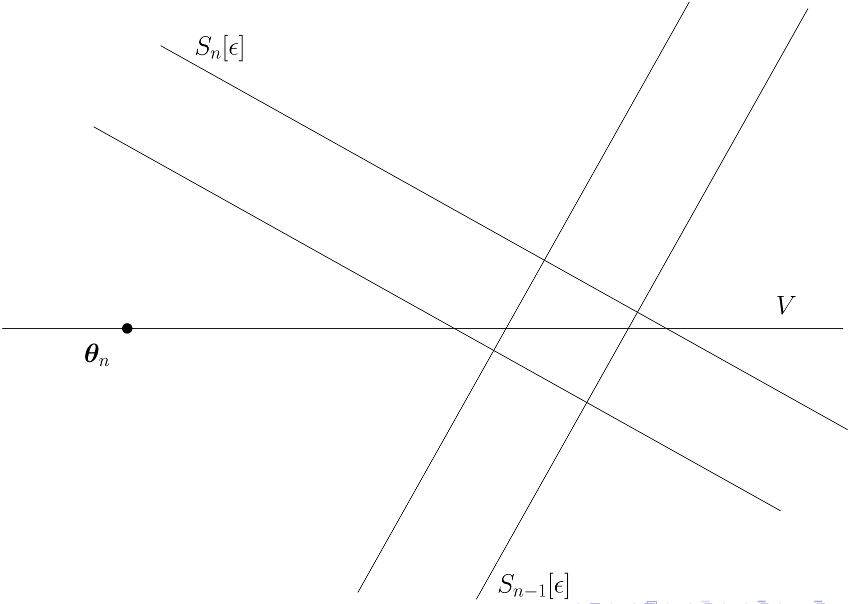

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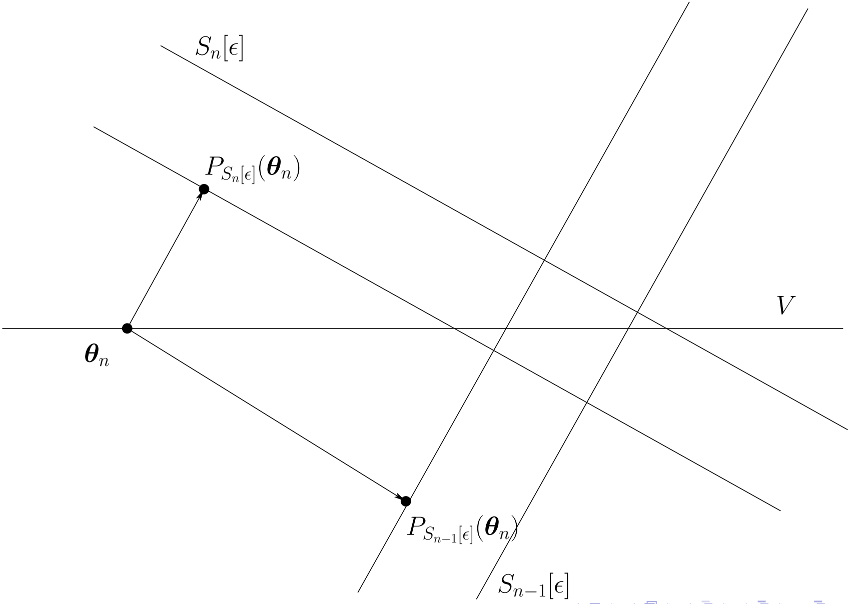

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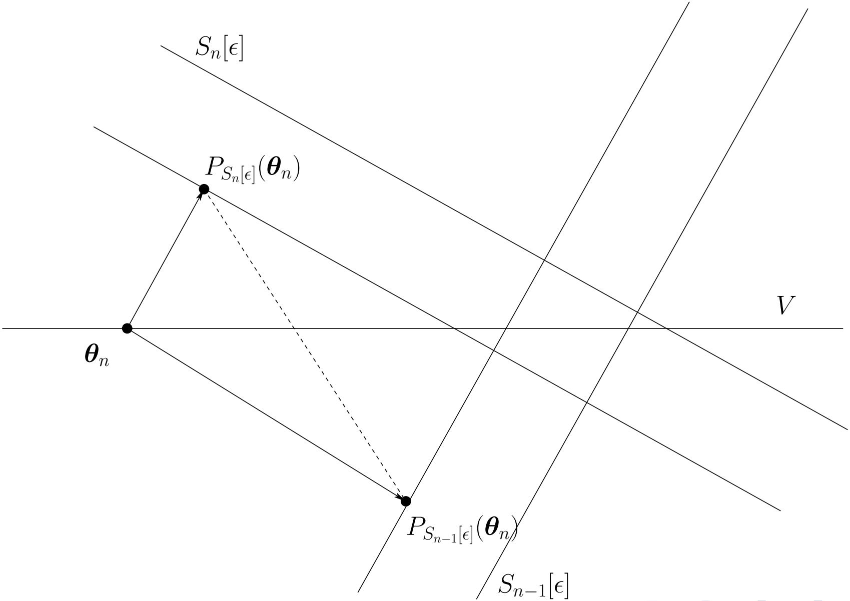

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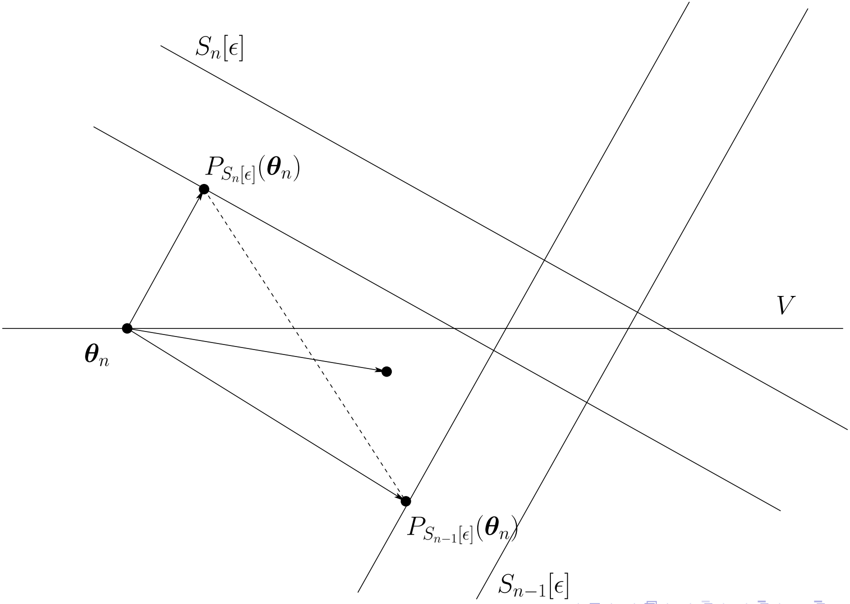

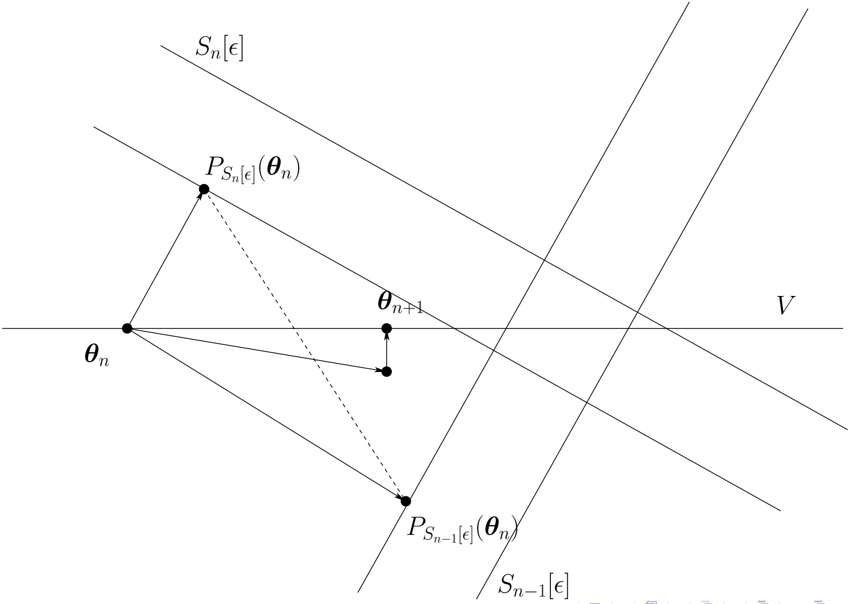
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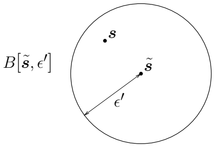

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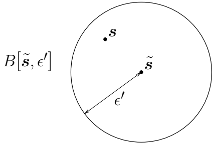


  - calculate those $\boldsymbol{\theta}$ such that, for some user-defined $\epsilon'' \geq 0$,

$$\boldsymbol{\theta}^t \boldsymbol{s} \in [1 - \epsilon'', 1 + \epsilon''], \quad \forall \boldsymbol{s} \in B[\tilde{\boldsymbol{s}}, \epsilon'].$$

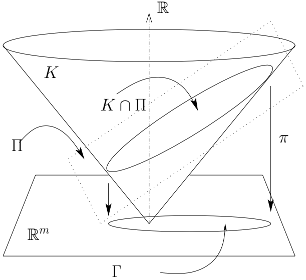
The Icecream Cone

- The previous task breaks down to a number of more fundamental problems of the following type; find a vector that belongs to

$$\Gamma := \{\boldsymbol{\theta} \in \mathbb{R}^m : \boldsymbol{\theta}^t \boldsymbol{s} \geq \gamma, \forall \boldsymbol{s} \in B[\tilde{\boldsymbol{s}}, \epsilon']\} = \left\{ \begin{array}{c} \text{all vectors that satisfy an} \\ \text{infinite number of inequalities} \end{array} \right\}.$$

- If $\Gamma \neq \emptyset$, then the previous problem is equivalent to[8]

finding a point in $K \cap \Pi$,
K: an icecream cone,
Π: a hyperplane.



[8] [Slavakis, Yamada' 07], [Slavakis, Theodoridis, Yamada '09].

The Complete Picture

Given $(\boldsymbol{x}_n, y_n)$, find a $\boldsymbol{\theta} \in \mathbb{R}^m$ such that

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Algorithm for Robust Regression

Assume weights $\omega_j^{(n)} \geq 0$ such that $\sum_{j=n-q+1}^{n} \omega_j^{(n)} = 1$. For any $\boldsymbol{\theta}_0 \in \mathbb{R}^m$,

$$\boldsymbol{\theta}_{n+1} := P_{\Pi} P_K \left(\boldsymbol{\theta}_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n \right) \right), \quad \forall n \geq 0,$$

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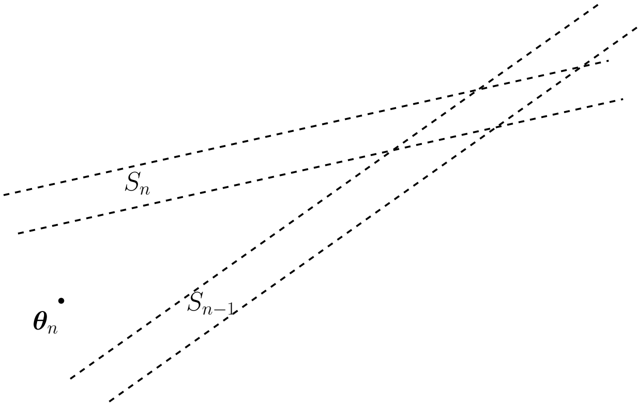

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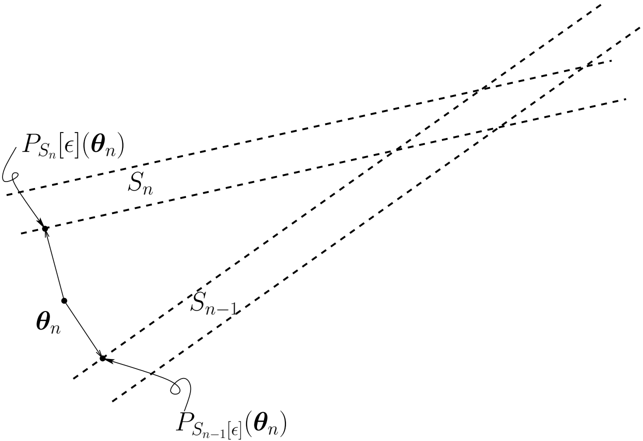

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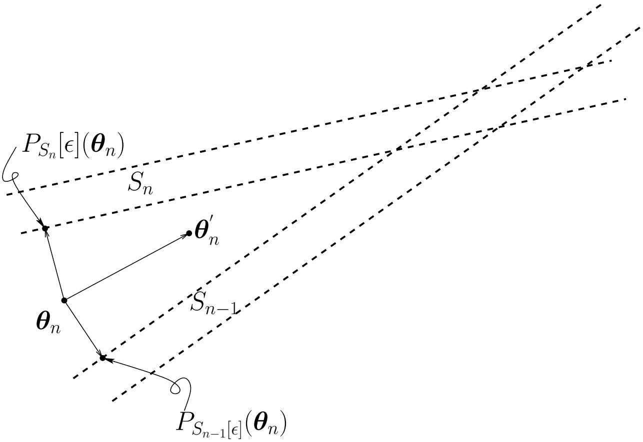

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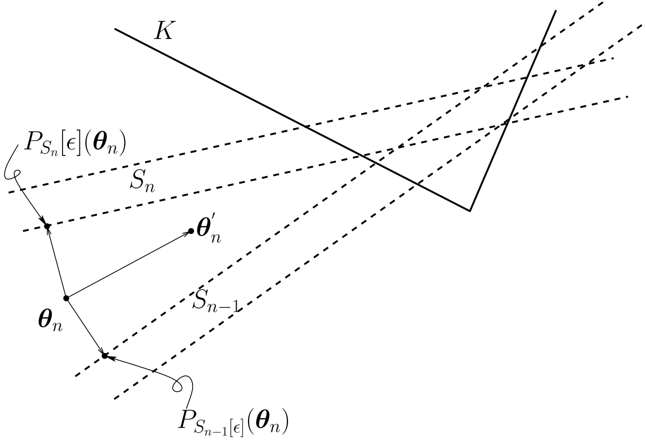

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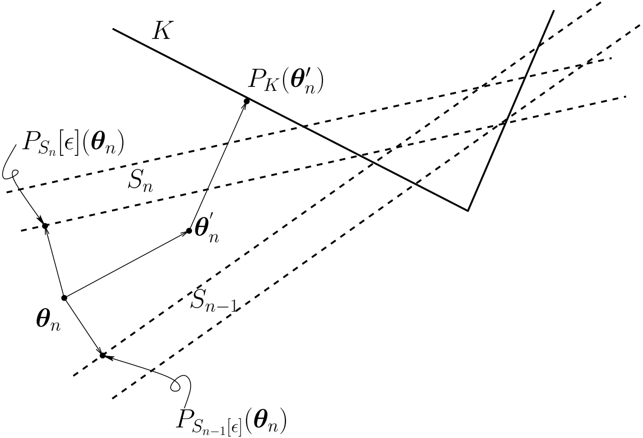

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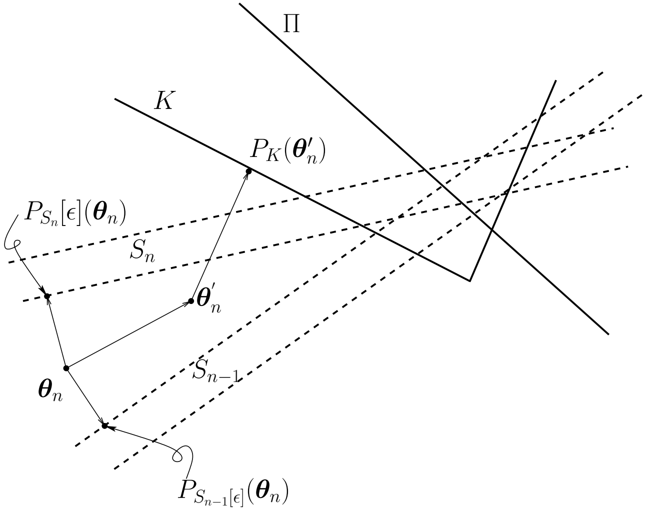

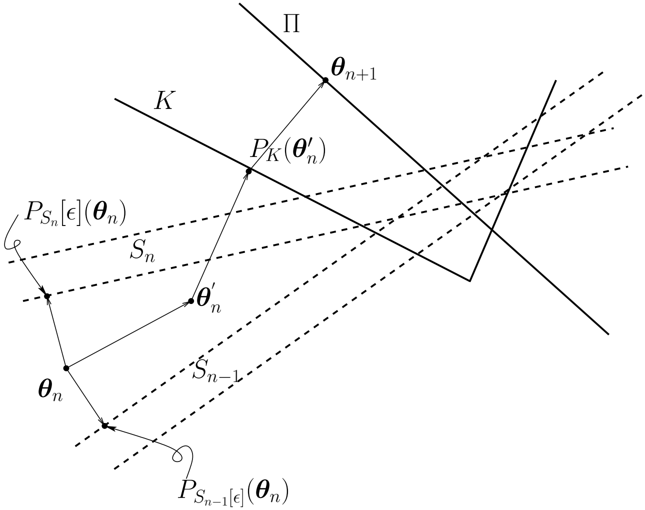
Π
$\boldsymbol{\theta}_{n+1}$
K
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This strategy reminds us of POCS:

POCS

Given a finite number of closed convex sets $C_1, \ldots, C_p$, with $\bigcap_{i=1}^p C_i \neq \emptyset$, let their associated projection mappings be $P_{C_1}, \ldots, P_{C_p}$. Then,

$$\forall \boldsymbol{\theta} \in \mathbb{R}^m, \quad \left(P_{C_p} \cdots P_{C_1}\right)^n (\boldsymbol{\theta}) \xrightarrow[n\to\infty]{w} {}^{\exists}\boldsymbol{\theta}_* \in \bigcap_{i=1}^p C_i.$$

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Key assumption

The a-priori info is consistent, i.e., $\bigcap_{i=1}^p C_i \neq \emptyset$.

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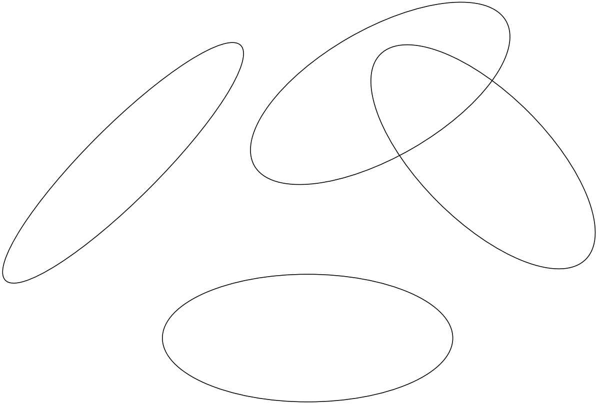
How do we deal with the case of inconsistent a-priori info, i.e.,

$$\bigcap_{i=1}^{p} C_i = \emptyset?$$

An Intuitive Suggestion



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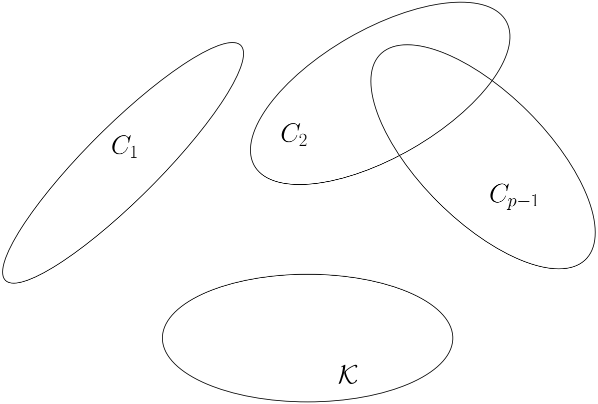

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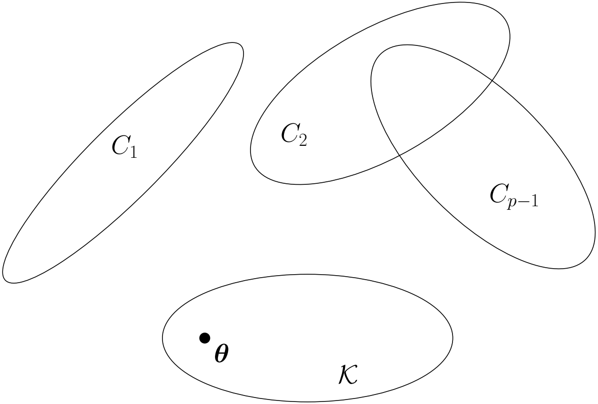

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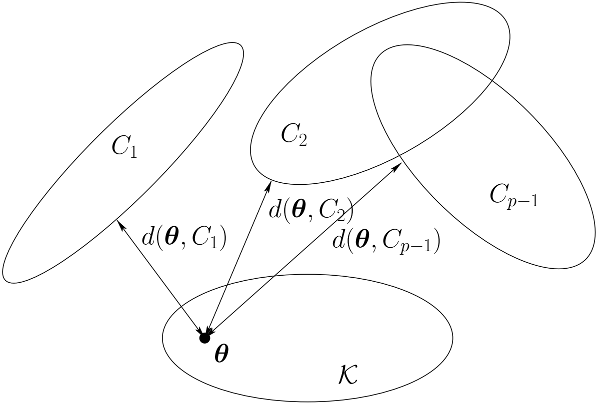

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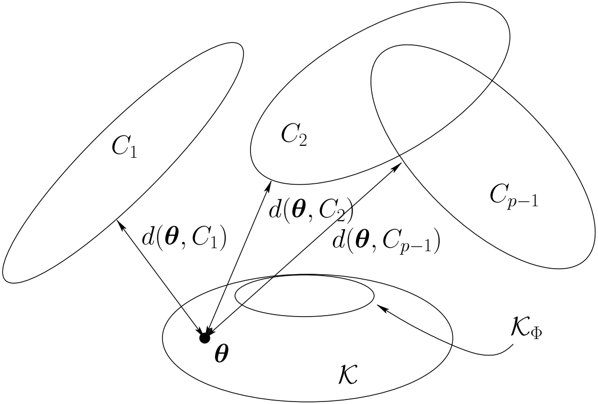

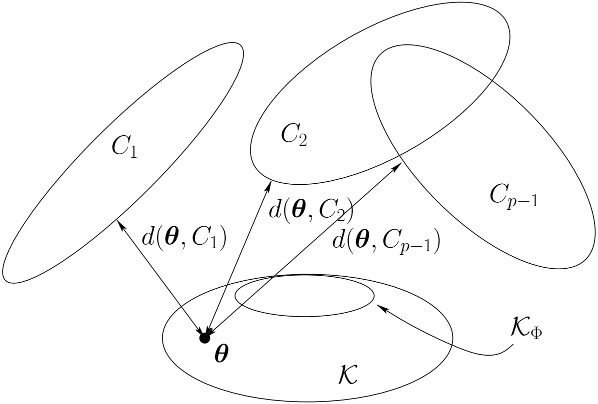

Definition ($\mathcal{K}_\Phi$)

All those points of $\mathcal{K}$ which minimize a function Φ of the distances $\{d(\cdot, C_i)\}_{i=1}^{p-1}$.

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- Then, $\mathrm{Fix}(T) = \mathcal{K}_\Phi$.

The Algorithm

For any $\boldsymbol{\theta}_0 \in \mathbb{R}^m$,

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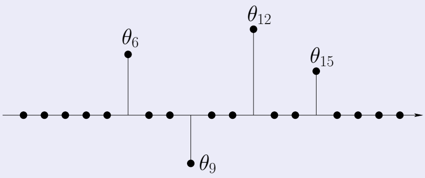
- Sparsity promotion is achieved via ℓ_1-norm regularization of a loss function:

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^m} \sum_{n=0}^{N} \mathcal{L}\left(y_n, \boldsymbol{x}_n^t \boldsymbol{\theta}\right) + \lambda \left\|\boldsymbol{\theta}\right\|_1, \quad \lambda > 0.$$

The ℓ_0 norm

$$\|\boldsymbol{\theta}\|_0 := \text{card}\{i : \theta_i \neq 0\}.$$



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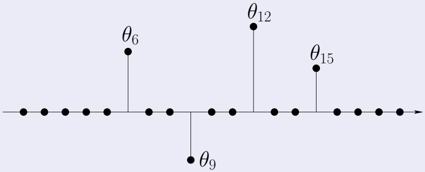

- Consider the linear model:

$$y_n := \boldsymbol{x}_n^t \boldsymbol{\theta} + v_n, \quad \forall n,$$

where $(v_n)_{n \geq 0}$ denotes the noise process.

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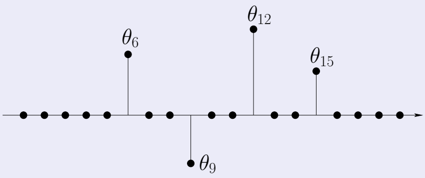

- Consider the linear model:

$$y_n := \boldsymbol{x}_n^t \boldsymbol{\theta} + v_n, \quad \forall n,$$

where $(v_n)_{n \geq 0}$ denotes the noise process.

- Define $\boldsymbol{X}_N := [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N]$, $\boldsymbol{y}_N := [y_0, y_1, \ldots, y_N]^t$, and $\epsilon \geq 0$.

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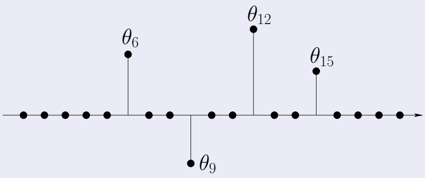

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- A typical Compressed Sensing task is formulated as follows:

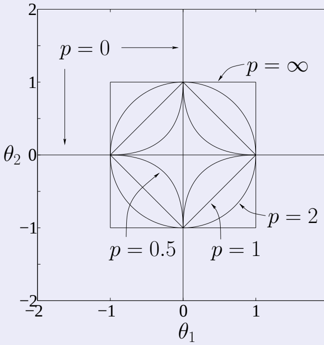
$$\min_{\boldsymbol{\theta} \in \mathbb{R}^m} \|\boldsymbol{\theta}\|_0$$

$$\text{s.t.} \quad \left\|\boldsymbol{X}_N^t \boldsymbol{\theta} - \boldsymbol{y}_N\right\| \leq \epsilon.$$

Alternatives to the ℓ_0 Norm

The ℓ_p norm $(0 < p \leq 1)$

$$\|\boldsymbol{\theta}\|_p := \left(\sum_{i=1}^{m} |\theta_i|^p \right)^{\frac{1}{p}}.$$



The ℓ_1-ball case

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- Given $(\boldsymbol{x}_n, y_n)$, $n = 0, 1, 2, \ldots$, find $\boldsymbol{\theta}$ such that

$$\left|\boldsymbol{\theta}^t \boldsymbol{x}_n - y_n\right| \leq \epsilon, \quad n = 0, 1, 2, \ldots$$
$$\boldsymbol{\theta} \in B_{\ell_1}[\delta] := \left\{\boldsymbol{\theta}' \in \mathbb{R}^m : \left\|\boldsymbol{\theta}'\right\|_1 \leq \delta\right\}.$$

Algorithm for Sparsity-Aware Learning

The ℓ_1-ball case

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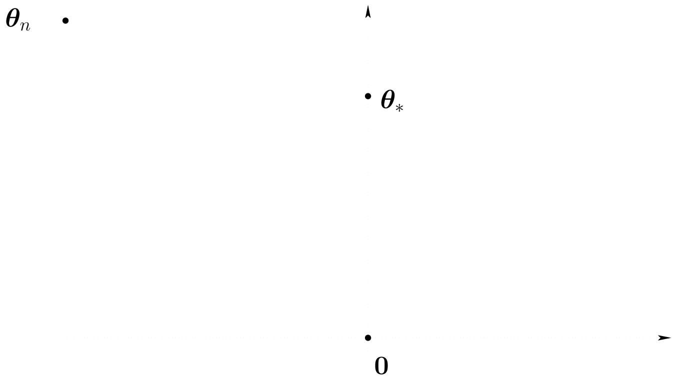
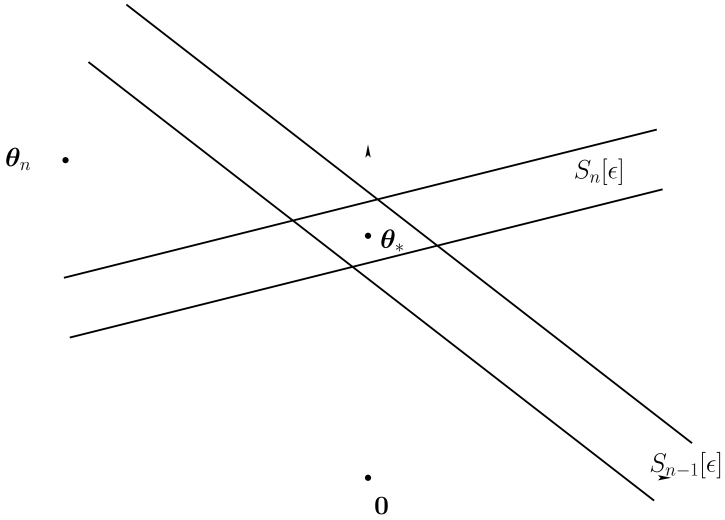
- The recursion:

$$\boldsymbol{\theta}_{n+1} := P_{B_{\ell_1}[\delta]}\left(\boldsymbol{\theta}_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{\theta}_n) - \boldsymbol{\theta}_n\right)\right).$$


$\boldsymbol{\theta}_n$
$\boldsymbol{\theta}_*$
$\mathbf{0}$

Geometric Illustration of the Algorithm



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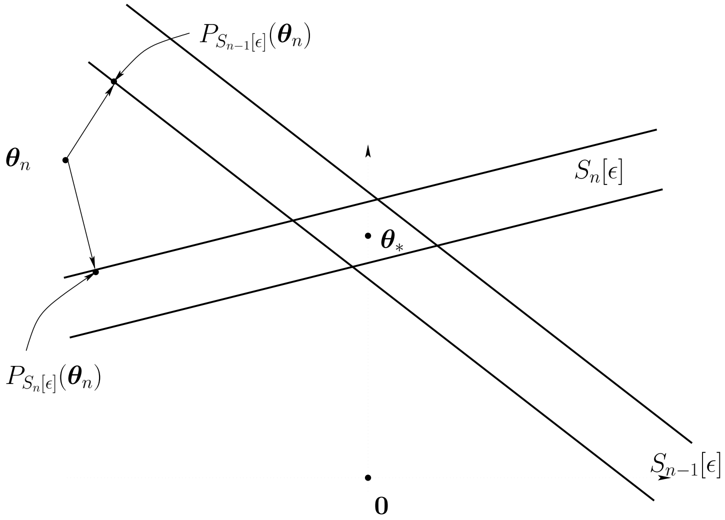

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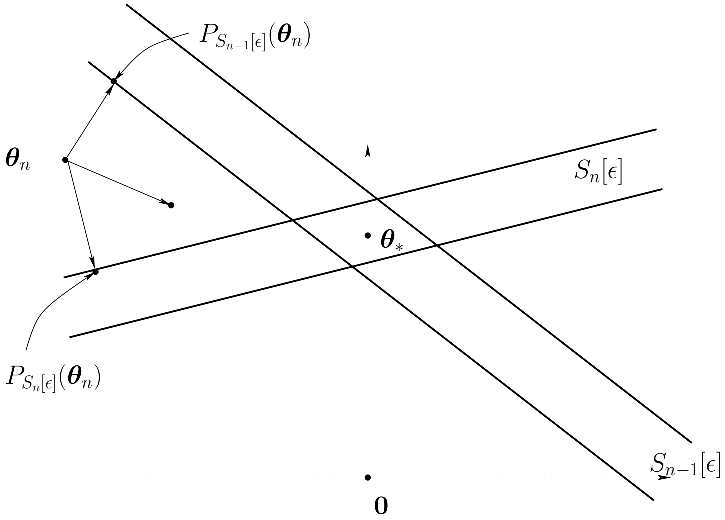

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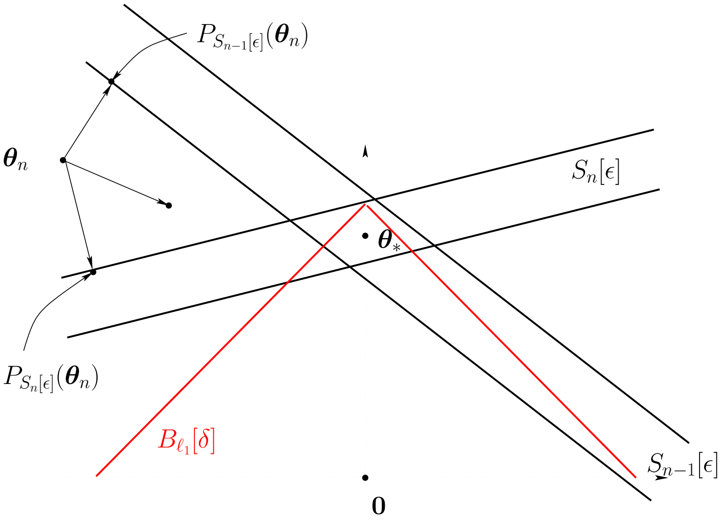

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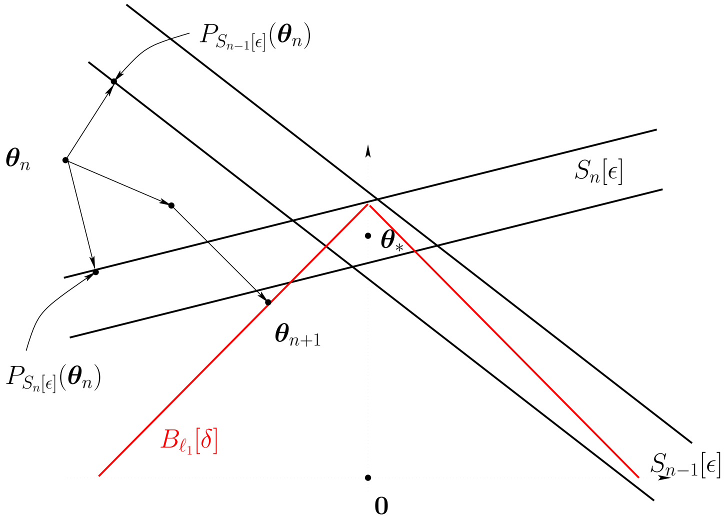

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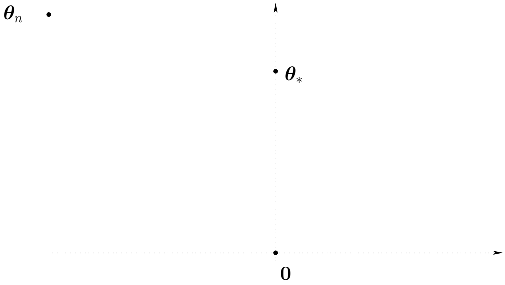
[9] [Kopsinis, Slavakis, Theodoridis, '11].


θ_n
θ_*
0

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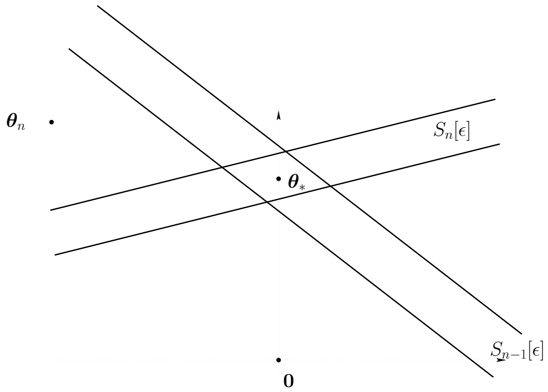

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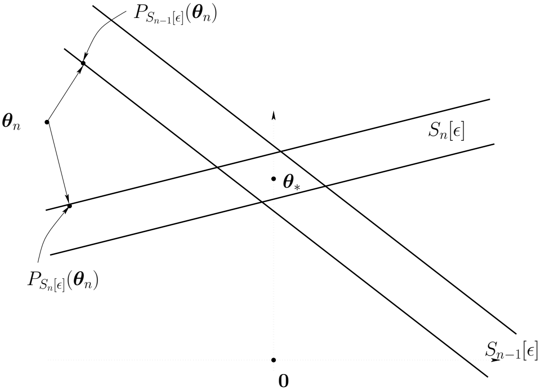

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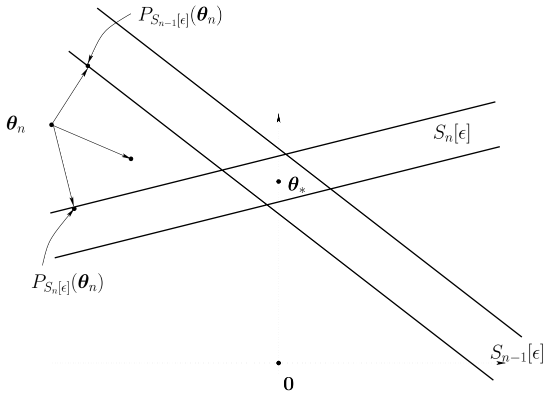

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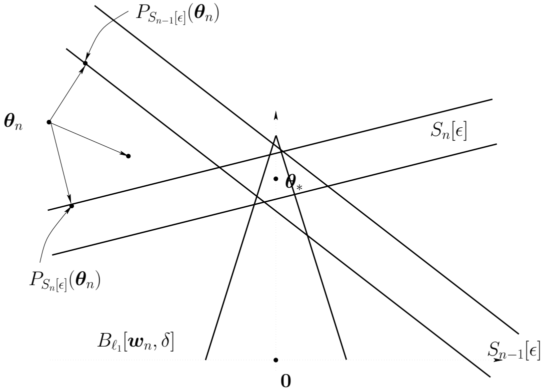

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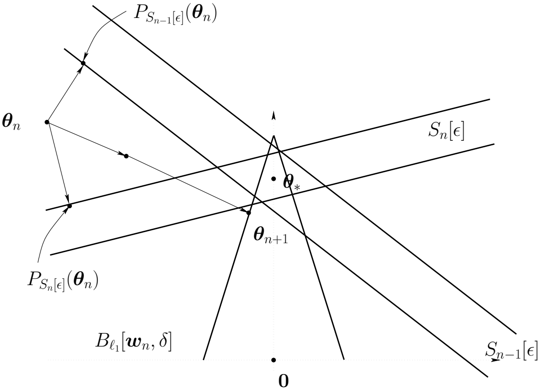

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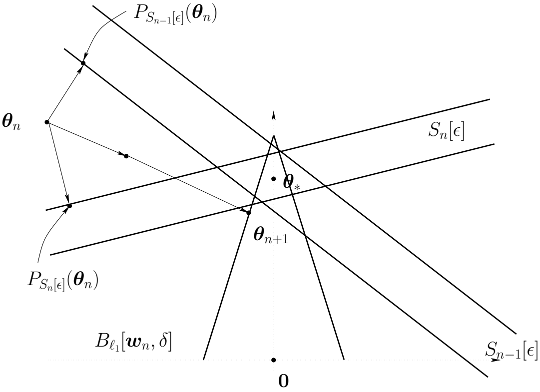

Projecting onto $B_{\ell_1}[\boldsymbol{w}_n, \delta]$ is equivalent to a specific soft thresholding operation.

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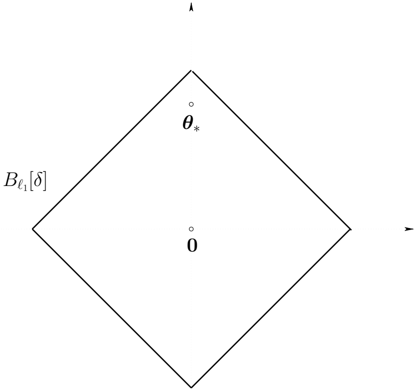

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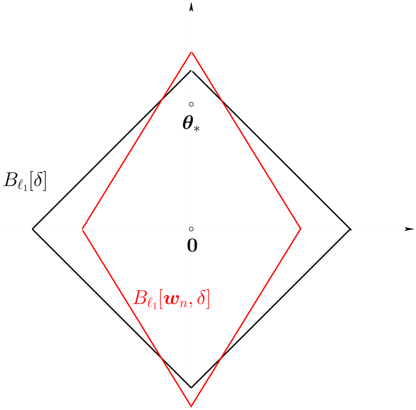

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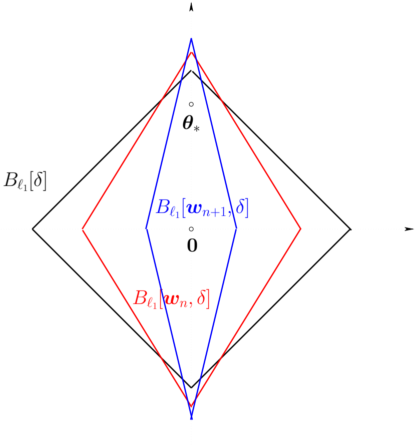
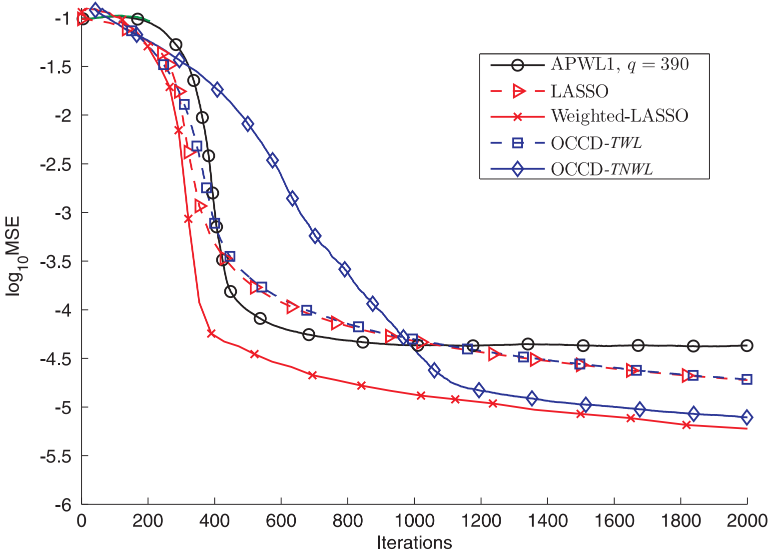
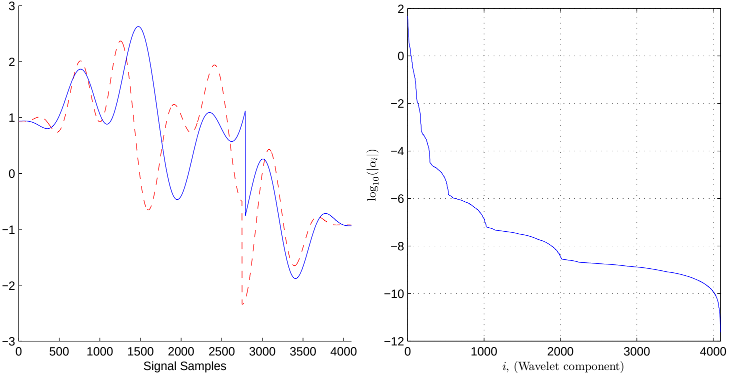


Time Invariant Signal



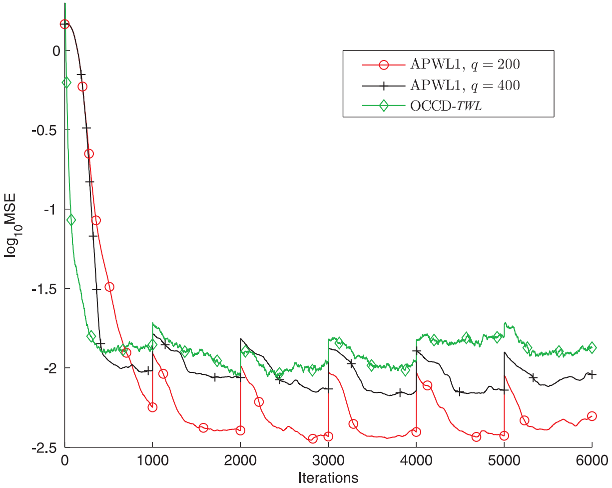
$m := 1024$, $\|\boldsymbol{\theta}_*\|_0 := 100$ wavelet coefficients. The radius of the ℓ_1-ball is set to $\delta := 101$.

Time Varying Signal



$m := 4096$. The radius of the ℓ_1-ball is set to $\delta := 40$.
The sum of two chirp signals.

Time Varying Signal



Movies of the OCCD, and the APWL1sub.

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What we have seen so far corresponds to soft thresholding operations.

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Hard thresholding

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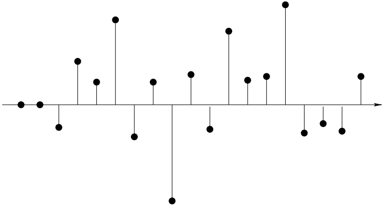

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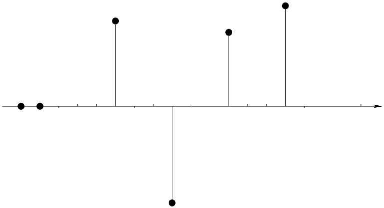

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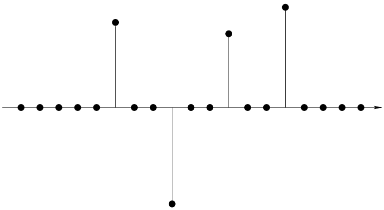

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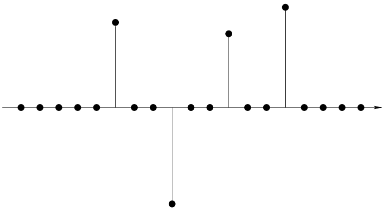

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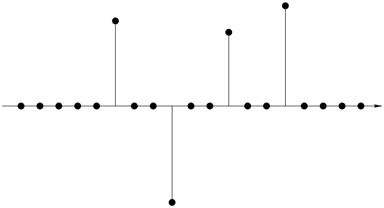
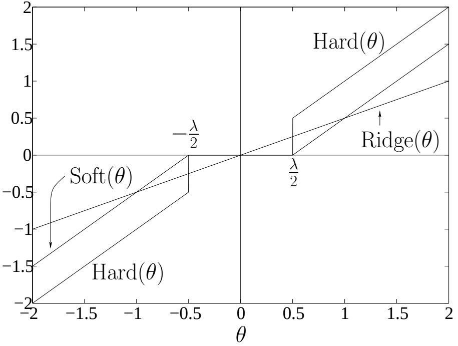


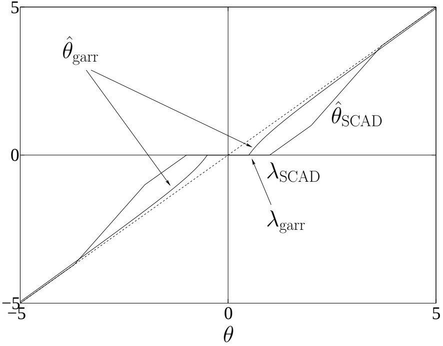
Generalized thresholding

- Identify the K largest, in magnitude, components of a vector $\boldsymbol{\theta}$.
- Shrink, under some rule, the rest of the components.

Examples of Generalized Thresholding Mappings



(a) Hard, soft thresholding, and the ridge regression estimate.



(b) The SCAD and garrote thresholding.

Penalized Least-Squares Thresholding

- Identify the K largest, in magnitude, components of a vector $\boldsymbol{\theta}$.

Mathematical Formulation of Thresholding

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where $p(\cdot)$ stands for a user-defined penalty function, which might be non-convex.

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Definition (Generalized Thresholding Mapping)

The Generalized Thresholding mapping is defined as follows:

$$T_{\mathrm{GT}} : \theta_i \mapsto \hat{\theta}_{i*}.$$

Fixed Point Set of T_{GT}

- Given K, define the set of all tuples of length K:

$$\mathscr{T} := \{(i_1, i_2, \ldots, i_K) : 1 \leq i_1 < i_2 < \ldots < i_K \leq m\}.$$

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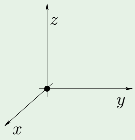
Example

For the 3-dimensional case $\mathbb{R}^3$, and if $K := 2$,

$$\mathrm{Fix}(T_{\mathsf{GT}}) = \begin{array}{c} xy\text{-plane} \cup yz\text{-plane} \\ \cup\, xz\text{-plane}. \end{array}$$



Definition (Nonexpansive Mapping)

A mapping $T : \mathcal{H} \to \mathcal{H}$ is called nonexpansive if

$$\|T(f_1) - T(f_2)\| \leq \|f_1 - f_2\|, \qquad \forall f_1, f_2 \in \mathcal{H}.$$

The fixed point set of a nonexpansive mapping is closed and convex.

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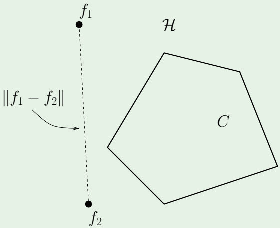

First Steps Towards a Unifying Framework

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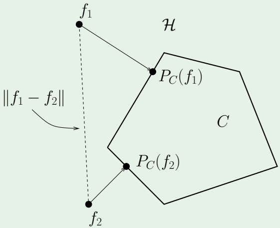

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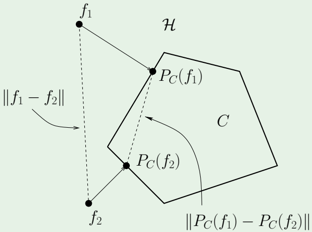


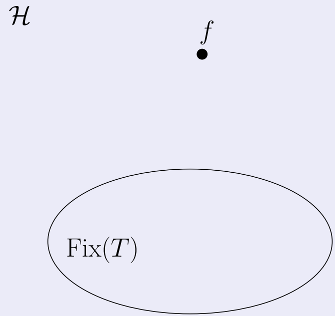
$$\mathrm{Fix}(P_C) = C.$$

Definition (Quasi-nonexpansive Mapping)

A mapping $T : \mathcal{H} \to \mathcal{H}$, with $\mathrm{Fix}(T) \neq \emptyset$, is called quasi-nonexpansive, if

$$\|T(f) - h\| \leq \|f - h\|, \quad \forall f \in \mathcal{H}, \forall h \in \mathrm{Fix}(T).$$

The fixed point set of T is convex.



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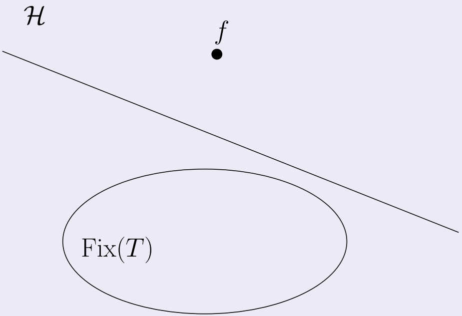

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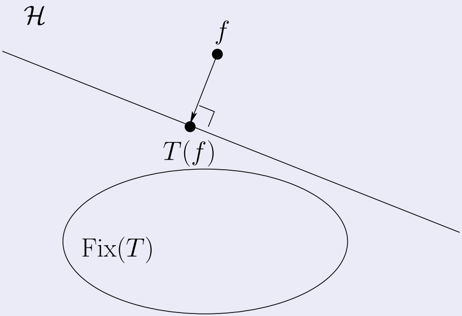

Quasi-nonexpansive Mapping

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A mapping $T : \mathcal{H} \to \mathcal{H}$, with $\mathrm{Fix}(T) \neq \emptyset$, is called quasi-nonexpansive, if

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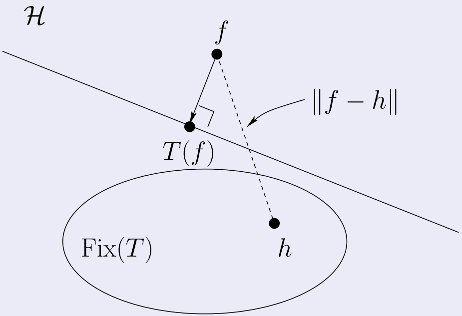

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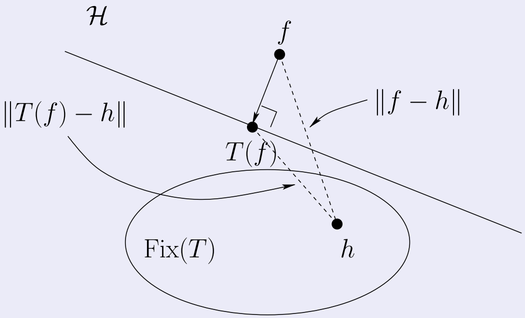

Every nonexpansive mapping is quasi-nonexpansive.

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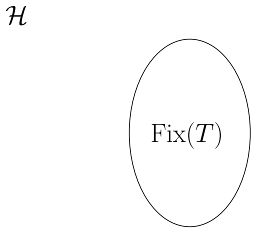

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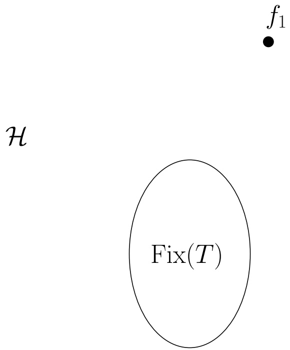

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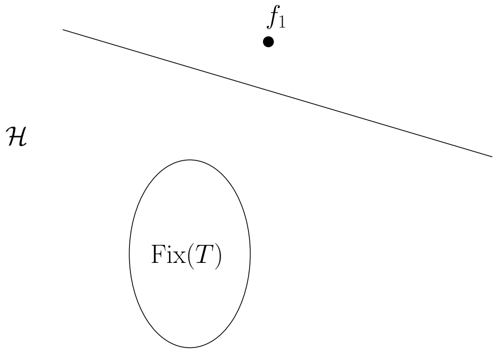

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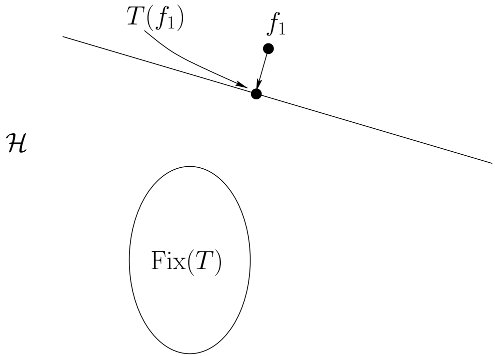

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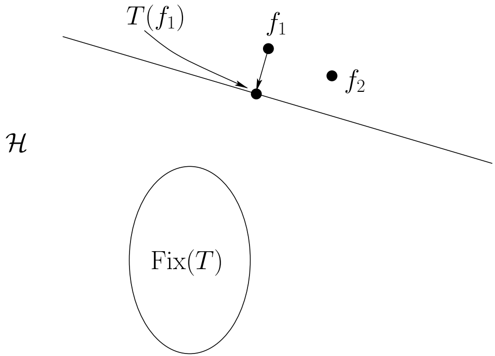

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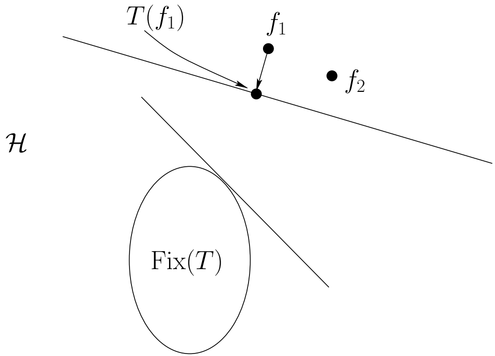

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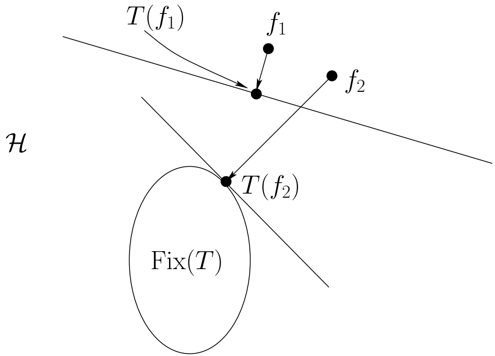

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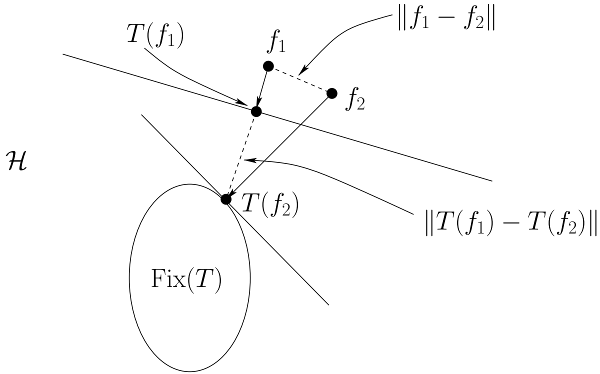

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In other words, the hyperplane $\big\{\big(g, \langle g - f, \Theta'(f)\rangle + \Theta(f)\big) : g \in \mathcal{H}\big\}$, supports the graph of Θ at the point $\big(f, \Theta(f)\big)$.

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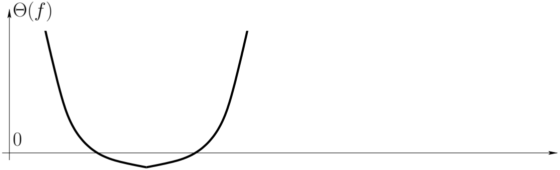

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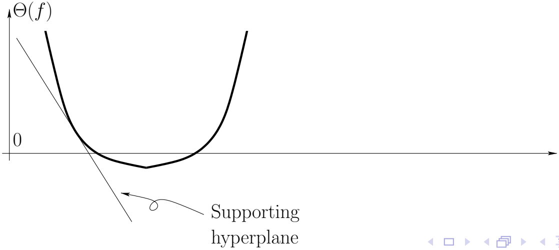

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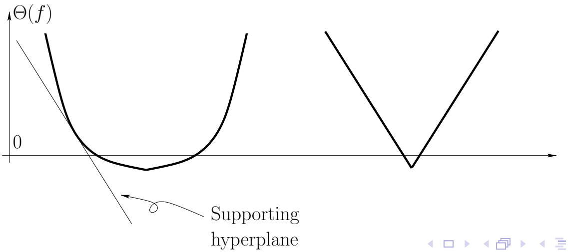

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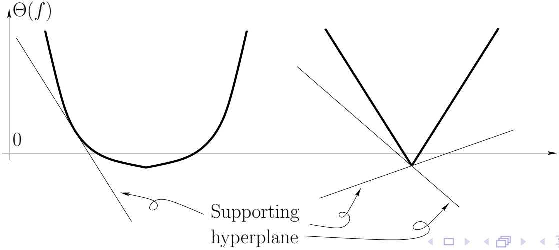

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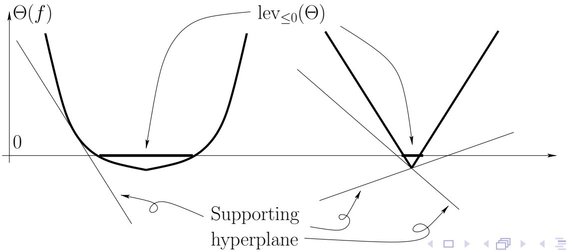

The Subgradient Projection Mapping

A Quasi-nonexpansive mapping

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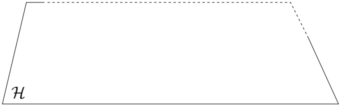

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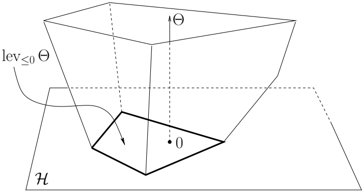

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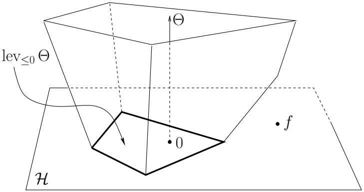

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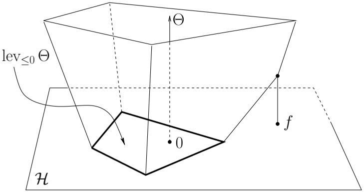

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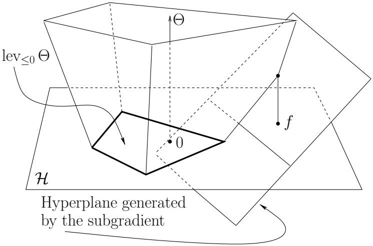

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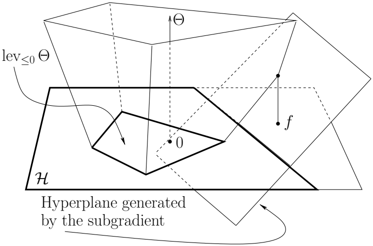

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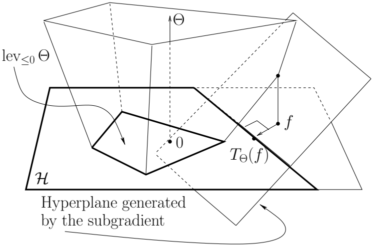

Generalizing the Previous Methodologies

Adaptive Projected Subgradient Method (APSM)

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$$f_{n+1} := \overbrace{\underbrace{\left(\alpha R_n + (1-\alpha)I\right)}_{=T_n:\ \text{averaged quasi-nonexpansive mapping}}}^{\text{a-priori info}} \overbrace{\underbrace{\left((1-\lambda)I + \lambda T_{\Theta_n}\right)}_{=\ \text{relaxed subgradient projection } T_{\Theta_n}}}^{\text{loss function info}} (f_n)$$

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- $(R_n)_{n=0,1,\ldots}$ is a sequence of quasi-nonexpansive mappings. This sequence of mappings comprises the a-priori information.
- $(\Theta_n)_{n=0,1,\ldots}$ is a sequence of loss/penalty function which quantifies the deviation of the sequential training data from the underlying model.

Candidates for the Loss Functions $(\Theta_n)_{n=0,1,\ldots}$

Given the current estimate f_n, define $\forall f \in \mathcal{H}$,

$$\Theta_n(f) := \begin{cases} \sum_{i=n-q+1}^{n} \frac{\omega_i^{(n)} d(f_n, S_i[\epsilon])}{\sum_{j=n-q+1}^{n} \omega_j^{(n)} d(f_n, S_j[\epsilon])} d(f, S_i[\epsilon]), & \text{if } f \notin \bigcap_{i=n-q+1}^{n} S_i[\epsilon], \\ 0, & \text{otherwise.} \end{cases}$$

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where the extrapolation coefficient $\mu_n \in (0, 2\mathcal{M}_n)$ with

$$\mathcal{M}_n := \begin{cases} \frac{\sum_{j=n-q+1}^{n} \omega_j^{(n)} \|P_{S_j[\epsilon]}(f_n) - f_n\|^2}{\|\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(f_n) - f_n\|^2}, & \text{if } \sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(f_n) \neq f_n, \\ 1, & \text{otherwise.} \end{cases}$$

Example (Examples of averaged quasi-nonexpansive mappings)

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Candidates for the Averaged Quasi-nonexpansives T_n

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Candidates for the Averaged Quasi-nonexpansives T_n

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- The composition $P_{\mathcal{K}}\Big(I-\lambda\big(I-\sum_{i=1}^{p-1}\beta_i P_{C_i}\big)\Big)$, $\lambda\in(0,2)$, where $\mathcal{K}\cap\left(\bigcap_{i=1}^{p-1}C_i\right)=\emptyset$, (beamforming).

The Trip Still Goes on and the Topic Still Grows...

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- Surprisingly, the APSM retains its performance and theoretical properties in the case where the Generalized Thresholding mapping T_{GT} is used in the place of T_n!
- Recall that $\mathrm{Fix}(T_{\mathrm{GT}})$ is a union of subspaces, which is a non-convex set.
- Such an application motivates the extension of the concept of a quasi-nonexpansive mapping to that of a partially quasi-nonexpansive one[10].

[10] [Kopsinis etal '11a].

Theoretical Properties

Define at $n \geq 0$, $\Omega_n := \mathrm{Fix}(T_n) \cap \mathrm{lev}_{\leq 0}\,\Theta_n$. Let $\Omega := \bigcap_{n \geq n_0} \Omega_n \neq \emptyset$, for some nonnegative integer n_0. Assume also that $\frac{\mu_n}{\mathcal{M}_n} \in [\epsilon_1, 2 - \epsilon_2]$, $\forall n \geq n_0$, for some sufficiently small $\epsilon_1, \epsilon_2 > 0$. Under the addition of some mild assumptions, the following statements hold true[11].

- Monotone approximation. $d(f_{n+1}, \Omega) \leq d(f_n, \Omega)$, $\forall n \geq n_0$.
- Asymptotic minimization. $\lim_{n\to\infty} \Theta_n(f_n) = 0$.
- Cluster points. If we assume that the set of all sequential strong cluster points $\mathfrak{S}\big((f_n)_{n=0,1,\ldots}\big)$ is nonempty, then

$$\mathfrak{S}\big((f_n)_{n=0,1,\ldots}\big) \subset \limsup_{n\to\infty} \mathrm{Fix}(T_n) \cap \limsup_{n\to\infty} \mathrm{lev}_{\leq 0}(\Theta_n),$$

  where $\limsup_{n\to\infty} A_n := \bigcap_{r>0} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \big(A_k + B[0,r]\big)$, and $B[0,r]$ is a closed ball of center 0 and radius r.
- Strong convergence. Assume that there exists a hyperplane $\Pi \subset \mathcal{H}$ such that $\mathrm{ri}_{\Pi}(\Omega) \neq \emptyset$. Then, there exists an $f_* \in \mathcal{H}$ such that $\lim_{n\to\infty} f_n = f_*$.

[11] [Slavakis, Yamada, '11].

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Matlab code

http://users.uop.gr/~slavakis/publications.htm