Novel Design of Equiripple Half-Band FIR Filters

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Abstract—A novel design procedure of equiripple half-band FIR filters is developed. Solution of the approximation problem in terms of generating function and zero phase transfer function for the equiripple half-band FIR filter is presented. The equiripple half-band FIR filters are optimal in the Chebyshev sense. The generating function for the equiripple half-band FIR filter is presented. The closed form solution provides an efficient computation of the impulse response of the filter. One example is included.

Keywords-FIR filter, half-band filter, equiripple approximation, optimal filter.

I. INTRODUCTION

Half-band (HB) filters are fundamental building blocks in multirate signal processing. They are used e.g. in filter banks and in compression techniques. The only available method for designing of equiripple (ER) HB FIR filters is based on the numerical McClellan - Parks program [1]. It is usually combined with a clever "Trick" [2]. Besides this, some design methods are available for almost ER HB FIR filters, e.g. [3], [4]. No general non-numerical design of an ER HB FIR filter was found in the references. In our paper we are primarily concerned with the ER approximation of HB FIR filters and with the related non-numerical design procedure suitable for practical design of ER HB FIR filters. We present the generating function and the zero phase transfer function of the ER HB FIR filter. These functions give an insight into the nature of this approximation problem. Based on the differential equation for the Chebyshev polynomials of the second kind, we have derived formulas for an effective evaluation of the coefficients of the impulse response. We present an approximating degree equation that is useful in practical filter design. The advantage of the proposed approach over the numerical design procedures consists in the fact that the coefficients of the impulse response are evaluated by formulas. Hence the speed of the design is deterministic.

II. IMPULSE RESPONSE, TRANSFER FUNCTION AND ZERO PHASE TRANSFER FUNCTION

An HB filter is specified by the minimal passband frequency $\omega_p T$ (or maximal stopband frequency $\omega_s T$) and by the minimal attenuation in the stopband a_s [dB] (or maximal attenuation in the passband a_p [dB]). The antisymmetric behavior of its frequency response implies the relations $\omega_s T = \pi - \omega_p T$ and $10^{0.05a_p} + 10^{0.05a_s} = 1$. The goal in the filter design is to get the minimum filter length N satisfying the filter specification and to evaluate the coefficients of the impulse response of the filter. We assume the impulse response $h(k)$ with odd length $N = 2(2n+1)+1$ coefficients and with even symmetry $h(k) = h(N-1-k)$. The impulse response of the HB FIR filter contains $2n$ zero coefficients as follows

$$\begin{aligned} &h(2n+1) = a(0) = 0.5 \\ &2h(2n+1 \pm 2k) = a(2k) = 0, k = 1...n \\ &2h(2n+1 \pm (2k+1)) = a(2k+1), k = 0...n \ . \end{aligned} \tag{1}$$

The transfer function of the HB FIR filter is

$$H(z) = z^{-(2n+1)} \left[\frac{1}{2} + \sum_{k=0}^{n} a(2k+1) T_{2k+1}(\omega) \right] \tag{2}$$

where $T_k(w)$ is Chebyshev polynomials of the first kind. The frequency response $H(e^{j\omega T})$ of the HB FIR filter is

$$H(e^{j\omega T}) = e^{-j(2n+1)\omega T} Q(\cos \omega T) \tag{3}$$

where $Q(w)$ is a polynomial in the variable $w = (z + z^{-1})/2$ which on the unit circle reduces to a real valued zero phase transfer function $Q(w)$ of the real argument $w = \cos(\omega T)$.

III. GENERATING POLYNOMIAL AND ZERO PHASE TRANSFER FUNCTION OF AN ER HB FIR FILTER

The generating polynomial of an ER HB FIR filter is related to the generating polynomial of the almost ER HB FIR filter presented in [4]. Based on our experiments conducted in [4], we have found that the generating polynomial $G(w)$ (Fig.1) of the ER HB FIR filter is obtained by weighting of Chebyshev polynomials in the generating polynomial of the almost ER HB FIR filter, namely

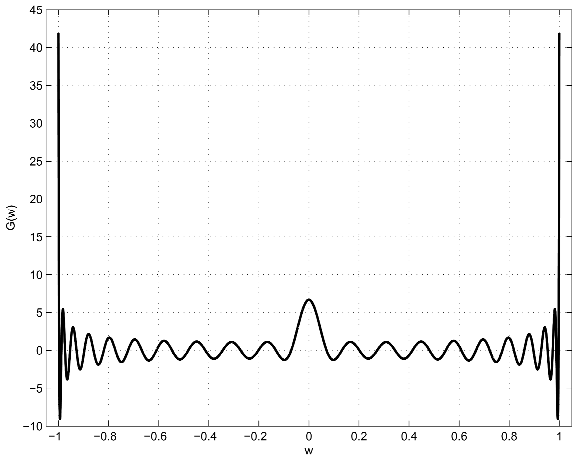

Figure 1. Generating polynomial $G(w)$ for n=20, $\kappa' = 0.03922835$, A=1.08532371 and B=0.95360863

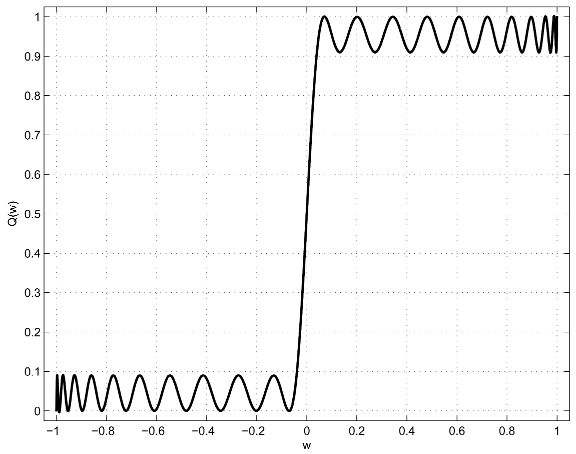

Figure 2. Zero phase transfer function $Q(w)$ for n=20, $\kappa' = 0.03922835$, A=1.08532371, B=0.95360863 and $\tilde{N} = 0.55091994$

$$G(w) = AU_n\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) + BU_{n-1}\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) \tag{4}$$

where $U_n(x)$ and $U_{n-1}(x)$ are Chebyshev polynomials of the second kind and A, B, κ' are real numbers. The zero phase transfer function $Q(w)$ (Fig. 2) is related to the generating polynomial

$$Q(w) = \frac{1}{2} + \frac{1}{\tilde{N}}\int G(w)dw \tag{5}$$

where the norming factor $\tilde{N}$ is given by (17). Both the generating polynomial $G(w)$ and the zero phase transfer function $Q(w)$ show the nature of the approximation of an ER HB FIR filter.

IV. Differential Equation and Impulse Response of an ER HB FIR Filter

The Chebyshev polynomial of the second kind $U_x(w)$ fulfils the differential equation

$$(1 - x^2)\frac{d^2U_n(x)}{dx^2} - 3x\frac{dU_n(x)}{dx} + n(n+2)U_n(x) = 0\ . \tag{6}$$

Using substitution

$$x = \left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) \tag{7}$$

we get the differential equation (6) in the form

$$\begin{aligned} &w(w^2 - \kappa'^2)\left[(1 - w^2)\frac{d^2U_n(w)}{dw^2} - 3w\frac{dU_n(w)}{dw}\right] \\ &+ \left[\kappa'^2(1 - w^2) + 2w^2(1 - w^2)\right]\frac{dU_n(w)}{dw} \\ &+ 4w^3n(n+2)U_n(w) = 0. \end{aligned} \tag{8}$$

Based on the differential equation (8), we have derived the non-numerical procedure for the evaluation of the impulse response $h_k(k)$ corresponding to polynomial corresponding to polynomial $\tilde{U}_n(w)$

$$\tilde{U}_n(w) = \int U_n\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right)dw. \tag{9}$$

This procedure is summarized in Tab.1. The impulse response $h(k)$ of the ER HB FIR filter is

$$h(k) = \frac{1}{2} + \frac{A}{\tilde{N}}h_n(k) + \frac{B}{\tilde{N}}h_{n-1}(k). \tag{10}$$

The non-numerical evaluation of the impulse response $h(k)$ is essential in the practical filter design.

V. Degree of an ER HB FIR Filter

The exact degree formula is not available. In the practical filter design, the degree n can be obtained with excellent accuracy from the specified minimal passband frequency ω_pT and from the minimal attenuation in the stopband a_s [dB] using the approximating degree formula

$$n \ge \frac{a_s[dB] - 18.18840664\omega_pT + 33.64775300}{18.54155181\omega_pT - 29.13196871}. \tag{11}$$

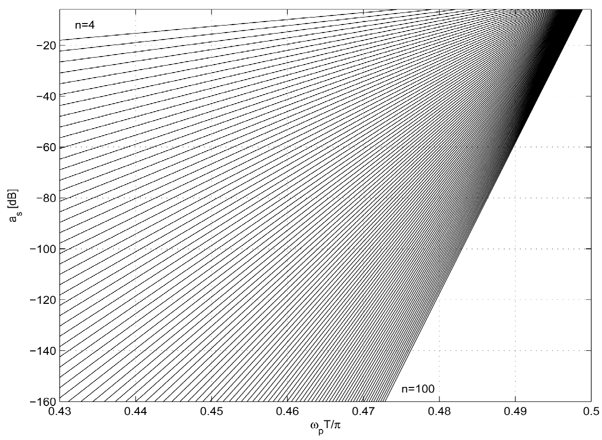

Figure 3. Emperical dependence of a_s [dB] on ω_pT/π and n

TABLE I. ALGORITHM FOR THE EVALUATION OF THE COEFFICIENTS $h_n(k)$

given	n (integer value), κ' (real value)	
initialization	$\alpha(2n) = \dfrac{1}{(1-\kappa'^2)^n}$	
	$\alpha(2n-2) = -(2n\kappa'^2+1)\,\alpha(2n)$	
	$\alpha(2n-4) = -\dfrac{4n+1+(n-1)(2n-1)\kappa'^2}{2n}\,\alpha(2n-2) - \dfrac{(2n+1)(n+1)\kappa'^2}{2n}\,\alpha(2n)$	
body (*for* $k=n$ *down to* 3)	$\alpha(2k-6) = \{ -\left[3(n(n+2)-k(k-2))+2k-3+2(k-2)(2k-3)\kappa'^2\right]\alpha(2k-4) - \left[3(n(n+2)-(k-1)(k+1))+2(2k-1)+2k(2k-1)\kappa'^2\right]\alpha(2k-2) - [n(n+2)-(k-1)(k+1)]\,\alpha(2k) \} \,/\, [n(n+2)-(k-3)(k-1)]$	
(*end loop on* k)		
integration		
(*for* $k=0$ *to* n)	$a(2k+1) = \dfrac{\alpha(2k)}{2k+1}$	(*end loop on* k)
impulse response $h_n(k)$	$h_n(2n+1) = 0$	
(*for* $k=0$ *to* n)	$h_n(2n+1\pm(2k+1)) = \dfrac{a(2k+1)}{2}$	(*end loop on* k)

The exact relation between the minimal attenuation in the stopband a_s [dB], the minimal passband frequency $\omega_p T$ and the degree n were obtained experimentally. It is shown in Fig.3. Equation (11) was obtained by the approximation of exact experimental values in Fig.3.

VI. SECONDARY VALUES OF THE ER HB FIR FILTER

The secondary real values κ', A and B can be obtained from the specified passband frequency $\omega_p T$ and from the degree n of the generating polynomial. In practical filter design, the approximating formulas

$$\kappa' = \frac{n\omega_p T - 1.57111377n + 0.00665857}{-1.01927560n + 0.37221484} \qquad (12)$$

$$A = \left(0.01525753n + 0.036823440 + \frac{9.24760314}{n}\right)\kappa' + 1.01701407 + \frac{0.73512298}{n} \qquad (13)$$

$$B = \left(0.00233667n - 1.35418408 + \frac{5.75145813}{n}\right)\kappa' + 1.02999650 - \frac{0.72759508}{n} \qquad (14)$$

obtained experimentally provide excellent accuracy. Further, the exact values κ', A and B can be obtained numerically (e.g. using the Matlab function *fminsearch*) by satisfying the equality (see Fig.4)

$$Q(w_p) = \begin{cases} Q(1) & \text{if } n \text{ is odd} \\ Q(w_{01}) & \text{if } n \text{ is even.} \end{cases} \qquad (15)$$

The value

$$w_{01} = \sqrt{\kappa'^2 + (1-\kappa'^2)\cos^2\frac{\pi}{2n+1}} \qquad (16)$$

was introduced in [4]. Relation (15) guarantees the equiriple behaviour of the generating polynomial $Q(w)$.

VII. DESIGN OF THE ER HB FIR FILTER

The design procedure is as follows:

- Specify the ER HB FIR filter by the minimal passband frequency $\omega_p T$ and by the minimal attenuation in the stopband a_s [dB].
- Calculate the integer degree n of the generating polynomial (11).
- Calculate the real values κ' (12), A (13) and B (14).
- Evaluate the partial impulse responses $h_n(k)$ and $h_{n-1}(k)$ (Tab.1).
- Evaluate the final impulse response $h(k)$ (10) where the real norming factor $\tilde{N}$ is

$$\tilde{N} = \begin{cases} 2\left[A\tilde{U}_n(1) + B\tilde{U}_{n-1}(1)\right] & \text{if } n \text{ is even} \\ 2\left[A\tilde{U}_n(\omega_{01}) + B\tilde{U}_{n-1}(\omega_{01})\right] & \text{if } n \text{ is odd.} \end{cases} \qquad (17)$$

VIII. EXAMPLE

Design the ER HB FIR filter specified by the minimal passband frequency $\omega_p T = 0.45\pi$ and by the minimal attenuation in the stopband $a_s = -120dB$.

We get $n = 38.3856 \rightarrow 39$ (11), $\kappa' = 0.15571103$ (12), A = 1.17117396 (13), B = 0.83763199 (14) and $\tilde{N}$ = -2747.96038544 (17). The impulse response $h(k)$ (Tab.2) with the length N=159 coefficients is evaluated using Tab.1 and (10). The actual values $\omega_{p\,act}T = 0.4502\pi$ and $a_{act} = -120.91$ dB satisfy the filter specification. The amplitude frequency response $20\log\left|H(e^{jwT})\right|$ [dB] of the filter is shown in Fig.5. The detailed view of its passband is shown in Fig.6. For the specified values $\omega_p T = 0.45\pi$ and N=159, the comparative numerical design based on the "Trick" [2]

TABLE II. COEFFICIENTS OF THE IMPULSE RESPONSE

k	h(k)	k	h(k)
0 , 158	-0.00000070	42 , 116	0.00231877
2 , 156	0.00000158	44 , 114	-0.00283354
4 , 154	-0.00000331	46 , 112	0.00344038
6 , 152	0.00000622	48 , 110	-0.00415347
8 , 150	-0.00001087	50 , 108	0.00498985
10 , 148	0.00001799	52 , 106	-0.00597048
12 , 146	-0.00002852	54 , 104	0.00712193
14 , 144	0.00004363	56 , 102	-0.00847897
16 , 142	-0.00006481	58 , 100	0.01008867
18 , 140	0.00009384	60 , 98	-0.01201717
20 , 138	-0.00013287	62 , 96	0.01436125
22 , 136	0.00018446	64 , 94	-0.01726924
24 , 134	-0.00025161	66 , 92	0.02098117
26 , 132	0.00033779	68 , 90	-0.02591284
28 , 130	-0.00044697	70 , 88	0.03285186
30 , 128	0.00058370	72 , 86	-0.04348979
32 , 126	-0.00075311	74 , 84	0.06223123
34 , 124	0.00096097	76 , 82	-0.10523903
36 , 122	-0.00121375	78 , 80	0.31802058
38 , 120	0.00151871	79	0.50000000
40 , 118	-0.00188398		

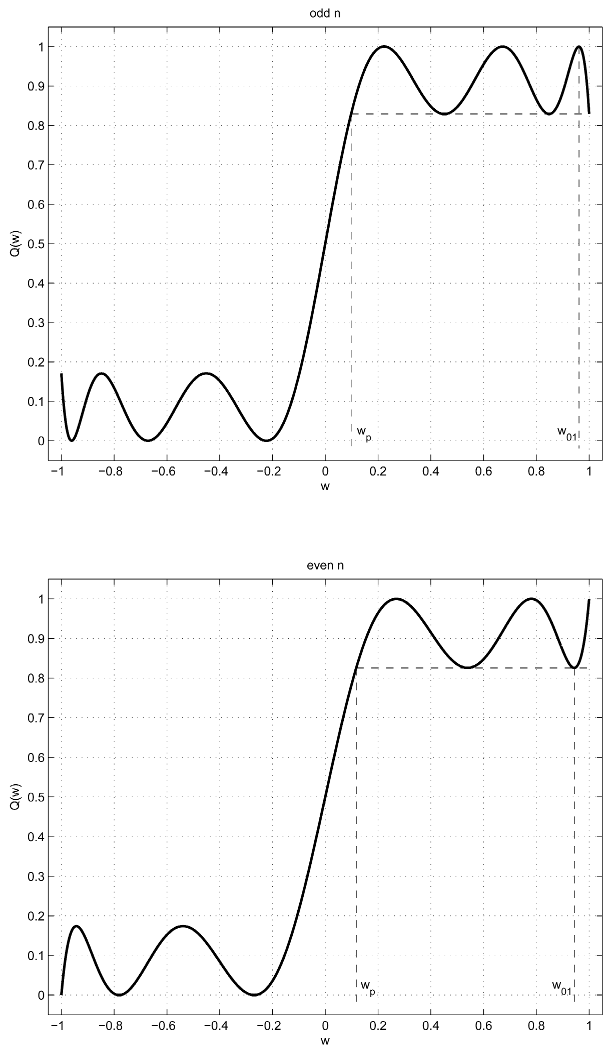

Figure 4. $Q(\omega)$ for odd and even n

combined with the Remez algorithm using the Matlab function *firpm* results in the slightly unsatisfactory minimal passband frequency $\omega_{p\ act}T = 0.44922001\pi < 0.45\pi$ and consequently in a slightly better minimal attenuation in the stopband $a_{act} = -123.29066608$ [dB].

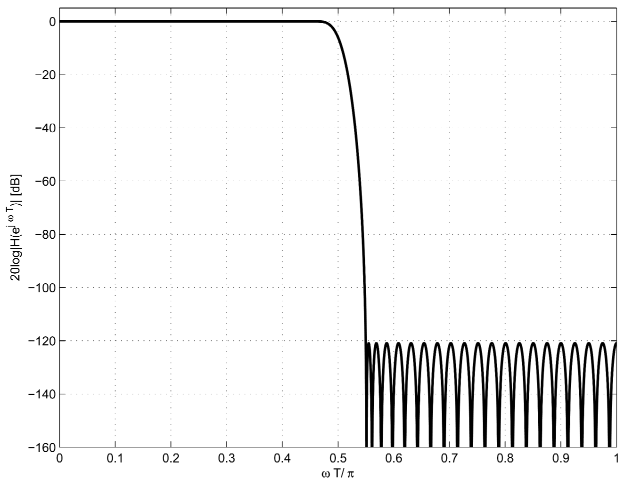

Figure 5. Amplitude frequency response $20\log\left|H(e^{j\omega T})\right|$

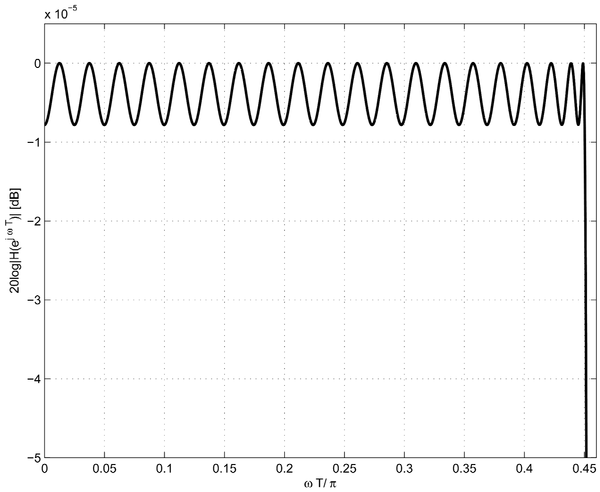

Figure 6. Passband of the filter

IX. CONCLUSIONS

This paper has presented the equiripple approximation of halfband FIR filters and a novel efficient non-numerical method for their design.

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